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1978 J. Phys. A: Math. Gen. 11 2405

(http://iopscience.iop.org/0305-4470/11/12/008)

J. Phys. A: Math. Gen., Vol. 11, No. 12, 1978. Printed in Great Britain

A new formulation of the field equations for the stationary axisymmetric vacuum gravitational field II. Separable solutions

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Received 10 April 1978, in final form 20 June 1978

Abstract. The techniques of the preceding paper are applied to several cases where the γ equation may be solved by separation of variables in the form, $\gamma=\gamma_1(\rho)+\gamma_2(\tau)$, where $\gamma_1(\rho)$ is either zero or a very simple function and $\gamma_2(\tau)$ satisfies an ordinary differential equation of the fourth order. Among the exact solutions constructed are the full six-parameter family of generalised Tomimatsu–Sato solutions, the rotating Curzon solution, the Kinnersley–Kelley solution and a class of solutions recently found by Ernst. Two new classes of solutions are presented as well as several new particular solutions expressible in closed form. In § 4, all stationary axisymmetric vacuum metrics with a non-trivial second-rank Killing tensor whose components do not depend on the ignorable co-ordinates, ϕ and t, are derived. This problem reduces to finding separable solutions of the dual of the γ equation of the form, $e^{2\gamma-2u}=R(\rho,\tau)[f(\rho)+g(\tau)]$, in four special co-ordinate systems, (ρ,τ), where $R(\rho,\tau)$ is a prescribed simple function. A comparison tis made with the canonical Schrödinger separable metric forms of Carter.

1. Introduction

In the preceding paper (Cosgrove 1978a, to be referred to as I, with equation numbers denoted by the prefix I, for example, I(2.7)), Einstein's equations for the stationary axisymmetric vacuum gravitational field were reformulated so that γ became the basic field variable and u, ω and the Ernst potentials became derived quantities. The functions, $\gamma=\gamma(r,z)$, $u=u(r,z)$ and $\omega=\omega(r,z)$, arise in the canonical form of the metric,

$$ds^2=e^{2u}(dt-\omega\,d\phi)^2-e^{-2u}\{e^{2\gamma}(dr^2+dz^2)+r^2\,d\phi^2\}, \tag{1.1}$$

and the Ernst potentials, ψ, $\mathscr{E}$ and ξ, are defined by I(1.7) and I(1.8). The field equation for γ is the fourth-order partial differential equation I(1.1) and the relationship of A, B, C and J to u and ψ is given by equations I(2.1a–e). Actually, it was found convenient to formulate the general theory in terms of arbitrary co-ordinates, $\rho=\rho(r,z)$ and $\tau=\tau(r,z)$ (note, particularly, the square-bracket-subscript notation in equations I(2.6a–e)), and in (ρ,τ) co-ordinates, the γ equation is I(2.7) with A, B, C and J given by I(2.9a–d). The rather non-trivial problem of constructing u and ψ from a given γ satisfying I(2.7) was solved in § 3 of I by three methods and found to depend on a pair of non-coupled ordinary differential equations.

In this paper, we focus attention on several particular co-ordinate systems, (r,z), (ρ,θ), (x,y), (ν,η), (σ,τ) and (α,β), which are defined in the appendix, and (s,λ) which is defined by equation (3.2*b*) below. In § 2 and § 3, we seek vacuum metrics for which γ takes the form,

$$\gamma = \gamma_1(\rho) + \gamma_2(\tau), \tag{1.2}$$

in four cases, $(\rho, \tau) = (\nu, \eta)$, (s, λ), (ρ, θ) and (z, r), the second and fourth cases yielding new solutions. The functional form of $\gamma_1(\rho)$ is restricted to depend on one parameter only so that $\gamma_2(\tau)$ satisfies a single ordinary differential equation of the fourth order and is therefore a transcendental function involving five parameters (one of which is a trivial additive constant). The first case, $(\rho, \tau) = (\nu, \eta)$, gives rise to the six-parameter generalised Tomimatsu–Sato solutions (Cosgrove 1977a, to be referred to as C, with equation numbers denoted by the prefix C), and our starting point in the derivation is a slight generalisation of 'Rule (*a*)' of Tomimatsu and Sato (1973). The remaining three cases give rise to limiting or contracted forms of the generalised Tomimatsu–Sato solutions and contain as special cases the rotating Curzon solution (Cosgrove 1977b), the extreme Kerr solution, the solution of Kinnersley and Kelley (1974), one of Lewis' solutions (Lewis 1932, pp 184–5) and an interesting four-parameter family of solutions due to Ernst (1977). We also give a number of new particular cases expressible in closed form.

In § 4, we search for all vacuum metrics of the form (1.1) possessing a non-trivial second-rank Killing tensor whose components do not depend on the ignorable coordinates, ϕ and t, making no *a priori* assumptions about separability of the Hamilton–Jacobi equation or about the algebraic type of the Riemann tensor. We are soon led to essentially the same canonical metric form as Carter (1968) who sought metrics for which the Hamilton–Jacobi and Schrödinger equations were separable in a certain way, but we employ the dual of the γ equation in order to pick out the vacuum metrics. The required solutions are found among the variable-separable solutions of the dual of the γ equation of the four forms,

$$e^{2\gamma-2u} = \frac{f(x)+g(y)}{x^2-y^2}, \qquad \frac{f(\rho)+g(\theta)}{\rho^2}, \qquad \frac{f(\sigma)+g(\tau)}{\sigma+\tau}, \qquad f(r)+g(z). \tag{1.3}$$

When we extend the argument to metrics not expressible in Lewis' form (1.1), we notice four metrics which are Hamilton–Jacobi and Schrödinger separable but which do not appear on Carter's list (Carter 1968, equations (4)–(19)). We also produce several examples which do not have the correct Lorentz signature of -2 including some that are not Hamilton–Jacobi separable.

2. Derivation of the six-parameter generalised Tomimatsu–Sato solutions

Let us construct the generalised Tomimatsu–Sato (TS) solutions starting from a suitable modification of TS 'Rule (*a*)' (TS 1973). The paper, C, did not provide a derivation from first principles but simply presented the results with proofs that they satisfied Einstein's vacuum equations and that a three-parameter subclass (containing the Kerr and TS series of rational function solutions) was asymptotically flat. The derivation given here is still not the original derivation from transformation groups (Cosgrove 1978b) but this present pair of papers will be a prerequisite for the planned future paper containing the original derivation.

Now Tomimatsu and Sato (1973) write the complex Ernst potential ξ as the ratio of two complex polynomials in x and y, i.e., $\xi = \alpha/\beta$, and write down seven 'Rules for Computation' labelled (a) to (g) satisfied by α and β. Some of these rules only apply to the original TS series of solutions. However, as pointed out in § 6 of C, the TS 'Rule (a)' may be written as the partial differential equation,

$$B_{[x,y]} \equiv 4u_x u_y + e^{-4u}\psi_x\psi_y = 0, \tag{2.1}$$

the notation $B_{[x,y]}$ being taken from I. There it was shown that $B_{[x,y]} = 0$ for all generalised TS solutions with the parameter $h = 0$. When $h \neq 0$, the rule must be generalised to

$$B_{[x,y]} = 4h(x^2-1)^{-1}(1-y^2)^{-1}. \tag{2.2}$$

(At the end of this section, we shall give formulae for α and β separately for the full six-parameter family and so give a more precise statement of TS 'Rule (a)'.)

We shall take, as starting point in our derivation, equation (2.2) with $h = h(\eta)$, an undetermined function of η, but will soon be forced to conclude that h is a constant (this was promised at the end of § 6 of C but, however, a claim made there that this assumption allows solutions outside the generalised TS class is erroneous). It is convenient to work with field variables, $A_{[x,y]}$, $B_{[x,y]}$, $C_{[x,y]}$ and $J_{[x,y]}$, even though the preferred co-ordinates are (ν, η). Hence, using equations (A.3) and (A.7b) from the appendix and (2.2), we find

$$\gamma_\nu = \frac{x^2(x^2-1)(1-y^2)}{2(x^2-y^2)} B_{[x,y]} = \frac{2h(\eta)}{1-\nu^2}. \tag{2.3a}$$

From (2.3a) we see that TS 'Rule (a)', i.e. $B_{[x,y]} = 0$, implies that γ is a function of η only. In the present case, (2.3a) implies that γ_η must take the form,

$$\gamma_\eta = h'(\eta)\ln\frac{1+\nu}{1-\nu} + \frac{k(\eta)}{2\eta(1+\eta)}, \tag{2.3b}$$

$k(\eta)$ arbitrary, and so from equations, (A.7a) and (A.7c) expressions for A, C and J are readily found. Substituting these into the field equation I(2.7) with $(\rho, \tau) = (x, y)$, we notice that the left hand side of I(2.7), when written as a function of η and ν, is a quartic polynomial in $\ln[(1+\nu)/(1-\nu)]$ whose coefficients are rational functions of ν. Clearly, if the left hand side of I(2.7) is to be identically zero, the coefficients of each power of $\ln[(1+\nu)/(1-\nu)]$ must vanish separately. Conveniently, the fourth-power term arises only from the term, $-4J^2$, in I(2.7). Thus the vanishing of the fourth-power term gives the following differential equation (DE) for $h(\eta)$,

$$[\eta(1+\eta)h'' + (1+2\eta)h'][(1+\eta)h'' + h'] = 0,$$

the prime denoting $d/d\eta$, whose general solution splits into two families,

$$h = \alpha\ln(1+1/\eta) + h_0 \qquad \text{and} \qquad h = \alpha\ln(1+\eta) + h_0, \tag{2.4}$$

α, h_0 constants. There is no loss of generality in considering the first case only as either case maps onto the other under the familiar symmetry, $(x, y) \to (y, x)$. But now B contains the logarithm term, $\ln(1+1/\eta)$, and A and C do not, so the coefficient of $\ln^4(1+1/\eta)$ in the left hand side of I(2.7) is easily seen to be $-1024\alpha^4(x^2-1)^{-4}(1-y^2)^{-4}$ thereby forcing α to vanish and reducing h to a constant.

Now, from (2.3a, b) and (A.7a, b, c), we have, explicitly

$$A_{[x,y]}=4(x^2-1)^{-2}(-\eta k'+k), \tag{2.5a}$$

$$B_{[x,y]}=4h(x^2-1)^{-1}(1-y^2)^{-1}, \tag{2.5b}$$

$$C_{[x,y]}=-4(1-y^2)^{-2}k', \tag{2.5c}$$

$$J_{[x,y]}=16(x^2-1)^{-2}(1-y^2)^{-2}(\eta k'^2-kk'-h^2). \tag{2.5d}$$

The field equation I(2.7) simplifies to

$$\begin{aligned}(1/256)(x^2-1)^4(1-y^2)^4\{\text{left hand side of I(2.7)}\}\\ =[-2\eta^2(1+\eta)k'''-2\eta(1+2\eta)k''][\eta k'^2-kk'-h^2]\\ +\eta^2(1+\eta)k''^2(2\eta k'-k)-4(\eta k'^2-kk'-h^2)^2=0.\end{aligned} \tag{2.6}$$

A first integral is readily found to be

$$\eta^2(1+\eta)^2k''^2=4[\eta k'^2-kk'-h^2][-(1+\eta)k'+k-\delta^2] \tag{2.7}$$

where δ is a constant.

With $k(\eta)$ replaced by $H_4(\eta)$, equation (2.7) is seen to be precisely the H_4 equation, C(3.14). Observe that (2.3a, b) with h constant correspond to C(6.9). As in C, we introduce functions, $\Gamma(\eta)$ and $\bar\Gamma(\eta)=(\text{constant})\eta^{\delta^2}\Gamma(\eta)$, according to

$$k=\delta^2+\eta(1+\eta)\Gamma'/\Gamma=-\delta^2\eta+\eta(1+\eta)\bar\Gamma'/\bar\Gamma \tag{2.8}$$

so that

$$e^{2\gamma}=\text{constant}\left(\frac{1+\nu}{1-\nu}\right)^{2h}\left(1+\frac{1}{\eta}\right)^{-\delta^2}\Gamma(\eta) \tag{2.9a}$$

$$=\text{constant}\left(\frac{1+\nu}{1-\nu}\right)^{2h}(1+\eta)^{-\delta^2}\bar\Gamma(\eta). \tag{2.9b}$$

The three-parameter asymptotically flat generalised TS solutions arise by setting $h=0$ in (2.7) and imposing the boundary condition,

$$k(\eta)\equiv H_4(\eta)=\delta^2p^{-2}+O(\eta^{-1}) \qquad \text{as} \qquad \eta\to\infty, \tag{2.10}$$

where p is a constant and $p^2+q^2=1$. The three parameters are κ (from (A.2)), q and δ and, with $\kappa=mp\delta^{-1}$, the solution represents the vacuum exterior of a gravitational source of mass m, angular momentum m^2q and quadrupole given by C(8.14c). Of course, with only $e^{2\gamma}$ determined so far, the full metric is not uniquely determined but, according to theorem 1 of § 3 of I, the full class of metrics is generated from a particular one by the SL(2) transformation group **P** defined by I(1.11). However, for the metric to be asymptotically Minkowskian and have $m>0$, the matrix representing I(1.11) is uniquely determined and we shall arrange for it to be the unit matrix, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)=(1, 0, 0, 1)$. Dropping this requirement, we have the four-parameter (generalised TS)–NUT metrics, counting only the NUT parameter λ, defined by I(1.12a), as an extra parameter, the others being trivial. In the asymptotically non-flat cases, the case of zero NUT parameter is not well defined. The fifth and sixth parameters of the full family are obtained, respectively, by dropping the boundary condition (2.10) and by restoring h to (2.7) and (2.9).

Further integration of equation (2.7) is not a straightforward matter. For the asymptotically flat cases, an extremely rapidly converging infinite series method is outlined in § 10 of C. In the TS cases, when δ is an integer, the series terminates so that Γ is a polynomial in η^{-1} of degree δ^2 and $k = H_4$ is a rational function of η (see § 11 of C). This series method may be adapted to the asymptotically non-flat cases including the $h \neq 0$ cases. The method generalises rather neatly to a five-parameter class (parameters κ, q, δ, h and λ) of solutions with $h \neq 0$ where an analogue of (2.10) holds: setting

$$\delta^2 = n^2 + 2mn + 2m^2, \qquad h = m(n+m), \tag{2.11}$$

the first two terms in the perturbation expansions of H_4 and $\bar{\Gamma}$ in the parameter q^2p^{-2}, $p^2+q^2=1$, are

$$H_4(\eta) = [(n+m)^2 - m^2\eta] + q^2p^{-2}\eta(1+\eta)V(\eta) + O(q^4p^{-4}), \tag{2.12}$$

$$\bar{\Gamma}(\eta) = p^{2n}\eta^{\delta^2}\Gamma(\eta)$$

$$= p^{2n}\eta^{(n+m)^2} - p^{2n-2}q^2\eta^{(n+m)^2}\int_\eta^\infty V(\mu)\,d\mu + p^{2n}O(q^4p^{-4}), \tag{2.13}$$

where

$$V = \frac{n(n+2m)}{(2m+1)^2}[\eta(1+\eta)W'^2 - (m^2 + (n+m)^2\eta^{-1})W^2], \tag{2.14}$$

$$W = \eta^{-m-1}\,{}_2F_1(n+2m+1, 1-n; 2m+2; -\eta^{-1}). \tag{2.15}$$

Note that, when $m = 0$ and $n = \delta$, the functions V and W here become identical to the functions V and W defined by C(10.20) and C(10.16a), respectively, and then (2.13) agrees with C(10.7), C(10.8a) and C(10.22a). As pointed out at the end of § 11 of C, there are two series of elementary functional solutions when $n = 1, 2, 3, \ldots$ and when $n+2m = 1, 2, 3, \ldots$. The cases $n = 1, 2$ and $n+2m = 1, 2$ are given explicitly by C(11.23) and C(11.24). The two series are not distinct, so consider only $n = 1, 2, 3, \ldots$ and put $\epsilon_4 = -1$ in C(11.23) and C(11.24). These elementary solutions arise because the power series for $\bar{\Gamma}$, whose first two terms are given by (2.13), terminates at the $(n+1)$th term, $(-1)^n 2q^{2n}\eta^{m^2}$. Most of the symmetry properties of the polynomials $\bar{\Gamma}$ for δ an integer described in § 11 of C, generalise neatly to the elementary functions $\bar{\Gamma}$ with n an integer and $m \neq 0$. In particular, TS 'Rule (c)' holds and is expressed by C(11.12). Below, we shall present the full metric in closed form for the case $n = 1$.

Now, most of the hard work in constructing the full metric for γ satisfying (2.3a, b) is done by the general theory of § 3 of I and we shall be able to compare formulae in § 3 of I directly with formulae in C. Define σ_1, σ_2, H_1 and H_2 by

$$\sigma_1 = \eta^{-1}(H_4 - \eta H_4')^{1/2}, \qquad \sigma_2 = (-H_4')^{1/2}, \tag{2.16a, b}$$

$$H_1 = \tfrac{1}{2}(\eta^2\sigma_1^2\sigma_2^2 - h^2)^{1/2}, \qquad H_2 = (-\delta^2 + \eta^2\sigma_1^2 + \sigma_2^2)^{1/2}, \tag{2.16c, d}$$

with signs of the square roots chosen so that the differential relations between these functions given by C(5.7a, b, c, e, f) hold. From (A.8a, b, c), we have

$$A_{[\nu,\eta]} = \frac{4(1+\eta)(\sigma_2^2 + \eta^2\sigma_1^2\nu^2 - 2h\nu)}{(1-\nu^2)^2(1+\eta\nu^2)},$$

$$B_{[\nu,\eta]} = \frac{-2H_4\nu + 2h(1-\eta\nu^2)}{\eta(1-\nu^2)(1+\eta\nu^2)},$$

$$C_{[\nu,\eta]}=\frac{\eta(\sigma_1^2+\sigma_2^2\nu^2)+2h\nu}{\eta(1+\eta)(1+\eta\nu^2)},$$

$$D_{[\nu,\eta]}=\frac{2H_1}{\eta(1-\nu^2)},$$

and, from I(3.10),

$$\Phi_{[\nu,\eta]}=\frac{8(1+\eta)}{(1-\nu^2)^3(1+\eta\nu^2)}[H_1(1-\nu^2)-H_2(\sigma_2^2+\eta^2\sigma_1^2\nu^2-2h\nu)].$$

By substituting these expressions for A and Φ into I(3.13), one constructs a third-order linear Fuchsian DE having

$$F_1=e^{-2u},\qquad F_2=-\psi\,e^{-2u},\qquad F_3=\psi^2\,e^{-2u}+e^{2u}$$

as linearly independent solutions. When $h=0$, this DE is precisely equation C(7.3) and may be simplified to C(7.6). When $h\neq 0$, the symmetry which permitted simplification to C(7.6) is not available, so the alternative $K^{(\epsilon)}$ equation, C(3.17) (see below), is more useful.

Taking $(\rho,\tau)=(\nu,\eta)$ and $\rho_0=\nu_0=0$, the functions, $M_0(\nu,\eta)$, $M(\nu,\eta)$, $\mu_0(\eta)$, $\mu_1(\eta)$ and $\mu_2(\eta)$, in § 3 of I are identical to the corresponding functions in § 4 and § 5 of C. Therefore the two Riccati equations, I(3.19) and I(3.18), are identical to the two Riccati equations, C(5.13), which reduce to C(4.15) and C(5.2), respectively, when $h=0$. The functions, V_0, W_0, V and W, of § 3 of I correspond, respectively, to I_0, J_0, I and J of §§ 4, 5 of C according to

$$V_0=\Lambda^{-1}I_0,\qquad W_0=\Lambda^{-1}J_0,\qquad V=\Lambda^{-1}I,\qquad W=\Lambda^{-1}J,$$

where $\Lambda=\mu_1^{-1}$.

When $h=0$ and $\rho_0=0$, the Riccati equation I(3.21) reduces to

$$d\mu/d\eta=(1+\eta)^{-1/2}\sigma_1\mu$$

allowing us to take the simple particular solution,

$$\mu_0\equiv 0,\qquad \mu_1=\Lambda^{-1}=\exp\left\{\int(1+\eta)^{-1/2}\sigma_1\,d\eta\right\},\qquad \mu_2\equiv 0. \tag{2.17a, b, c}$$

With this choice for μ_0, μ_1 and μ_2 in the asymptotically flat cases, the formulae I(3.26a, b, c) give those metrics which are free of the NUT singularity on the symmetry axis. Equation (2.17b) corresponds to C(3.6) and C(5.7d).

The function $L^{(\epsilon)}$ of § 3 of I is closely related to $K^{(\epsilon)}$ of C. Comparing equations I(3.28a, b) with C(5.20a, b), we see that the particular solutions, $L_1^{(\epsilon)}$ and $L_2^{(\epsilon)}$, of the $L^{(\epsilon)}$ equation I(3.17) are related to the particular solutions, $K_1^{(\epsilon)}$ and $K_2^{(\epsilon)}$, of the $K^{(\epsilon)}$ equation C(3.17) by

$$L_1^{(\epsilon)}=\mu_1^{-1/2}\theta^{-1}[(1+\nu)/(1-\nu)]^{\epsilon iH_2/2}K_1^{(\epsilon)}, \tag{2.18a}$$

$$L_2^{(\epsilon)}=\mu_1^{1/2}\theta^{-1}[(1+\nu)/(1-\nu)]^{\epsilon iH_2/2}K_2^{(\epsilon)}, \tag{2.18b}$$

where $\theta=(\eta\sigma_1\sigma_2)^{-1/2}(2H_1+\epsilon ih)^{1/2}$. Thus the boundary conditions C(5.21) on $K^{(\epsilon)}$, which reduce to C(3.9) when $h=0$, are easily deduced from the boundary conditions I(3.30) on $L^{(\epsilon)}$.

Finally, the formulae I(3.31a, b, c) for F_1, F_2 and F_3 become C(4.2a, b, c) and the formulae, C(4.17) and C(4.18), follow directly from I(3.26a, b, c). However, the formula, C(4.19) or C(4.20), for the metric coefficient ω is not easily derived from the general theory of § 3 of I and appears to be an accidental relationship although it arises quite naturally in the derivation in Cosgrove (1978b).

As an example, let us write down explicit closed-form expressions for the metric and Ernst potentials for the $h \neq 0$ generalisation of the Kerr metric. Take the parametrisation (2.11) with $n = 1$. Since h may take values in the range, $-\infty < h < \infty$, δ may be real or pure imaginary and m may be real (take $m \geq -\frac{1}{2}$) or complex with Re $m = -\frac{1}{2}$ (take Im $m > 0$). Consider, first, the case of real m. An expression for $e^{2\gamma}$ follows immediately from equation C(11.23) for $\bar{\Gamma}$. It is

$$e^{2\gamma} = b\left(\frac{1+\nu}{1-\nu}\right)^{2m(m+1)}(1+\eta)^{-2m(m+1)-1}[p^2\eta^{(m+1)2} - q^2\eta^{m2}] \tag{2.19a}$$

$$= b\frac{(x^2-1)^{m^2}(1-y^2)^{m^2}}{(x-y)^{(2m+1)^2}(x+y)}[p^2(x^2-1)^{2m+1} - q^2(1-y^2)^{2m+1}], \tag{2.19b}$$

b, q constants, $p^2 = 1 - q^2$. The solution C(11.23) for $\bar{\Gamma}$ was obtained from a Riccati equation for H_4 which is the condition that the solutions of the $K^{(\epsilon)}$ equation C(3.17) be free of logarithms at the regular singular point, $\nu = 1$, which has exponents, $-\frac{1}{2}$ and $\frac{1}{2}$. This same property allows the $K^{(\epsilon)}$ equation itself to be converted to a hypergeometric equation which may then be solved with elementary functions. The Riccati equation C(5.15), from which μ_0, μ_1 and μ_2 are constructed, does not reduce to a familiar standard equation but may be solved with elementary functions with a little ingenuity. We shall not work through these rather lengthy calculations here but the interested reader may construct any of these functions from the finished formulae for e^{2u}, ψ and ω.

The complex Ernst potential ξ is given by

$$\xi = \alpha/\beta,$$

where

$$\alpha = \tfrac{1}{2}p(x^2-1)^m[(x+1)^{m+1}(1-y)^m + (x-1)^{m+1}(1+y)^m] + \tfrac{1}{2}iq(1-y^2)^m[(x+1)^m(1-y)^{m+1} - (x-1)^m(1+y)^{m+1}], \tag{2.20a}$$

$$\beta = \tfrac{1}{2}p(x^2-1)^m[(x+1)^{m+1}(1-y)^m - (x-1)^{m+1}(1+y)^m] + \tfrac{1}{2}iq(1-y^2)^m[(x+1)^m(1-y)^{m+1} + (x-1)^m(1+y)^{m+1}]. \tag{2.20b}$$

Hence

$$e^{2u} = \left[\frac{(x-1)(1+y)}{(x+1)(1-y)}\right]^m \frac{p^2(x^2-1)^{2m+1} - q^2(1-y^2)^{2m+1}}{p^2(x^2-1)^{2m}(x+1)^2 + q^2(1-y^2)^{2m}(1-y)^2}, \tag{2.21a}$$

$$\psi = -\frac{2pq(x-1)^{2m}(1+y)^{2m}(x+y)}{p^2(x^2-1)^{2m}(x+1)^2 + q^2(1-y^2)^{2m}(1-y)^2}, \tag{2.21b}$$

$$\omega = -\frac{2\kappa qp(x+1)^{2m+1}(1-y)^{2m+1}(x+y)}{p^2(x^2-1)^{2m+1} - q^2(1-y^2)^{2m+1}}. \tag{2.21c}$$

Of course, in these formulae, we have chosen a particular NUT parameter and we are free to apply the transformation I(1.11). The chosen NUT parameter gives the neatest and simplest formulae for general m but, for $m = 0$, the above metric does not reduce to the Kerr metric proper but instead to the Kerr–NUT metric,

$$\xi=(p-\mathrm{i}q)(px-\mathrm{i}qy),$$

which has NUT parameter, $\lambda=-\sin^{-1} q$.

Clearly, if $h>-\frac{1}{4}$, then m is real and the metric is real-valued, but a real-valued metric is easily constructed for $h<-\frac{1}{4}$. Simply take p^2 and q^2 to be complex conjugate numbers with $\operatorname{Re} p^2=\operatorname{Re} q^2=\frac{1}{2}$, let b in (2.19) be pure imaginary and apply the transformation I(1.11) with $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)=2^{-1/2}\,(1, \mathrm{i}, \mathrm{i}, 1)$. If $q=0$, the metric is a Weyl static metric. A limiting form involving logarithms can be obtained from (2.21) when $m\to-\frac{1}{2}$. Another limiting form appears in § 3 below.

Finally, let us close this section by giving explicit formulae for α and β in the TS relation, $\xi=\alpha/\beta$, such that, in the cases where ξ is a rational function of x and y (e.g., the TS metrics and (2.20) with $m=0, 1, 2, 3, \ldots$), α and β are polynomials with no common factor and so give a more precise formulation of TS 'Rule (a)'. From C(4.2),

$$\mathscr{E}=-\mathrm{i}\mu_1\frac{K_2^{(+1)}}{K_1^{(+1)}},\qquad \xi=\frac{K_1^{(+1)}-\mathrm{i}\mu_1K_2^{(+1)}}{K_1^{(+1)}+\mathrm{i}\mu_1K_2^{(+1)}}. \tag{2.22a, b}$$

Thus, likely candidates are of the form,

$$\alpha=S(K_1^{(+1)}-\mathrm{i}\mu_1K_2^{(+1)}),\qquad \beta=S(K_1^{(+1)}+\mathrm{i}\mu_1K_2^{(+1)}), \tag{2.23}$$

where S is some function which cancels out common rational and non-rational factors in the numerator and denominator of (2.22b). A short calculation gives

$$\alpha\beta_x-\beta\alpha_x=-4\mu_1\sigma_1S^2(1-y^2)^{-1}, \tag{2.24a}$$

$$\alpha\beta_y-\beta\alpha_y=4\mu_1\sigma_1^{-1}S^2(x^2-1)^{-1}(2\mathrm{i}H_1-h). \tag{2.24b}$$

Thus $\alpha\beta_x-\beta\alpha_x$ is real and $\alpha\beta_y-\beta\alpha_y$ is pure imaginary in accordance with 'Rule (a)' if $h=0$ and S^2 is real. Now, when $h=0$ and δ is an integer, the polynomials α and β obeying all of the TS rules are given explicitly by (2.23) with

$$S=\tfrac{1}{2}(1-y^2)^{\delta^2/2}\Sigma_1 \tag{2.25}$$

where Σ_1 is defined by

$$\bar{\Gamma}=\Sigma_1\Sigma_2,\qquad \mu_1=\Sigma_2/\Sigma_1.$$

The TS–NUT metrics also obey 'Rule (a)' as the transformations C(3.13) leave the left hand sides of (2.24a, b) invariant. Now, when $h\neq0$, rational function solutions arise when n and m are both integers in the parametrisation (2.11) and, without loss of generality owing to symmetries, $n\geqslant1$ and $m\geqslant0$. In these cases, the natural splittings of μ_0, μ_1 and μ_2 into numerator and denominator are as follows:

$$\mu_0=\Sigma_3/\Sigma_1,\qquad \mu_1=\bar{\Gamma}/\Sigma_1^2,\qquad \mu_2=\Sigma_4/\Sigma_1,\qquad \mu_1+\mu_0\mu_2=\Sigma_2/\Sigma_1,$$
$$\bar{\Gamma}=\Sigma_1\Sigma_2-\Sigma_3\Sigma_4.$$

The factor S now takes the form,

$$S=k(x^2-1)^{-m(m-1)/2}(1-y^2)^{[(n+m)^2+m]/2}\Sigma_1\exp\left\{-\frac{1}{2}\mathrm{i}h\int\eta^{-2}(1+\eta)^{-1}\sigma_1^{-2}H_2\,\mathrm{d}\eta\right\}, \tag{2.26}$$

where k is a real or complex constant. This formula is in agreement with (2.25) when $h=0$ and with (2.20a, b) when $n=1$.

3. Contractions of the generalised Tomimatsu–Sato solutions

Three distinct families of metrics can be derived as contractions of the generalised TS solutions. First, invert the relations (A.2), replacing z by $z-z_0$:

$$x=(1/2\kappa)[r^2+(z-z_0+\kappa)^2]^{1/2}+(1/2\kappa)[r^2+(z-z_0-\kappa)^2]^{1/2}, \qquad (3.1a)$$

$$y=(1/2\kappa)[r^2+(z-z_0+\kappa)^2]^{1/2}-(1/2\kappa)[r^2+(z-z_0-\kappa)^2]^{1/2}. \qquad (3.1b)$$

The first case arises by putting $z_0=0$ and letting $\kappa\to 0$. We find (see appendix)

$$\kappa x\to\rho, \qquad y\to\cos\theta, \qquad (3.2a)$$

$$\kappa^2\eta\to\lambda\equiv\rho^2\operatorname{cosec}^2\theta, \qquad \kappa^{-1}\nu\to s\equiv\rho^{-1}\cos\theta. \qquad (3.2b)$$

The second case arises by putting $z_0=-\kappa$ and letting $\kappa\to\infty$. We find

$$2\kappa(x-1)\to\sigma, \qquad 2\kappa(1-y)\to\tau, \qquad (3.3a)$$

$$\eta\to(1+\cos\theta)/(1-\cos\theta), \qquad \kappa(1-\nu)\to\rho. \qquad (3.3b)$$

Alternatively, we could put $z_0=+\kappa$ and let $\kappa\to\infty$. We then find

$$2\kappa(x-1)\to\tau, \qquad 2\kappa(1+y)\to\sigma, \qquad (3.4a)$$

$$\eta\to(1-\cos\theta)/(1+\cos\theta), \qquad \kappa(1+\nu)\to\rho. \qquad (3.4b)$$

The third case arises by setting $z_0=0$ and letting $\kappa\to\infty$. We find

$$2\kappa^2(x-1)\to r^2, \qquad \kappa y\to z, \qquad (3.5a)$$

$$\kappa^2\eta\to r^2, \qquad \kappa\nu\to z. \qquad (3.5b)$$

3.1. *Contractions arising from* (3.2)

Comparing (2.8) and (2.9) with (3.2b), we seek separable solutions of the γ equation of the form,

$$\gamma=2\tilde{h}s+\frac{1}{2}\int\lambda^{-2}\tilde{k}(\lambda)\,\mathrm{d}\lambda, \qquad (3.6)$$

where $\tilde{h}$ is a constant. The field equation for γ reduces to a third-order DE for $\tilde{k}$ for which a first integral is readily found to be

$$\lambda^4\tilde{k}''^2=4(\lambda\tilde{k}'^2-\tilde{k}\tilde{k}'-\tilde{h}^2)(-\lambda\tilde{k}'+\tilde{k}-\tilde{\delta}^2), \qquad (3.7)$$

$\tilde{\delta}$ constant, the prime denoting $\mathrm{d}/\mathrm{d}\lambda$. Equation (3.7) is the limiting form as $\kappa\to 0$ of equation (2.7) if we set

$$h=\kappa^{-1}\tilde{h}, \qquad \delta=\kappa^{-1}\tilde{\delta}, \qquad k(\eta)=\kappa^{-2}\tilde{k}(\lambda). \qquad (3.8)$$

The metrics in this class involve five non-trivial parameters, four from (3.7) and the NUT parameter.

A two-parameter class of asymptotically flat solutions (mass m, angular momentum m^2q, quadrupole $\frac{1}{3}m^3(1+2q^2)$) occurs when we impose the conditions,

$$\tilde{h}=0, \qquad \tilde{\delta}=mp=m(1-q^2)^{1/2},$$
$$\tilde{k}(\lambda)=m^2+\mathrm{O}(\lambda^{-1}) \qquad \text{as} \qquad \lambda\to\infty, \qquad (3.9)$$

choosing the NUT parameter appropriately. This yields the 'rotating Curzon metric' which is constructed in greater detail in Cosgrove (1977b). It contains as special cases the static Curzon metric, where $q = 0$, and the extreme Kerr metric, where $q = \pm 1$, $\tilde{\delta} = 0$.

Since these metrics are so closely related to the generalised TS metrics, they are best studied by contracting formulae already derived for the generalised TS metrics using (3.2*a*, *b*) and (3.8) rather than from first principles. Since this procedure is quite straightforward, we shall not give further details. (*Note*: In the discussion of the rotating Curzon metric in Cosgrove (1977b), we used the scaling parameter δ rather than κ, recalling that $\kappa = mp\delta^{-1}$ in the asymptotically flat cases, so that the co-ordinates, λ and σ, and the function $\tilde{H}_4$ used there are equal, respectively, to $(mp)^{-2}\lambda$, mps and $(mp)^{-2}\tilde{k}$ in the present notation.)

A simple closed-form solution with $\tilde{h} = 0$ and $\tilde{\delta} = 0$ results when we contract the exact solution (2.19)–(2.21) by setting $\kappa = \beta p^{1/(2m+1)}$, β constant, and letting $p \to 0$. The limiting metric is precisely

$$e^{2\gamma} = (\text{constant})[\beta^{2m^2}\lambda^{-m^2} - \beta^{2(m+1)^2}\lambda^{-(m+1)^2}]$$

$$= (\text{constant})[(\beta\rho^{-1}\sin\theta)^{2m^2} - (\beta\rho^{-1}\sin\theta)^{2(m+1)^2}], \tag{3.10a}$$

$$e^{2u} = \left(\frac{1+\cos\theta}{1-\cos\theta}\right)^m \frac{(\rho/\beta)^{4m+2} - (\sin\theta)^{4m+2}}{(\rho/\beta)^{4m+2} + (1-\cos\theta)^2(\sin\theta)^{4m}}, \tag{3.10b}$$

$$\psi = -\frac{2(\rho/\beta)^{2m+1}(1+\cos\theta)^{2m}}{(\rho/\beta)^{4m+2} + (1-\cos\theta)^2(\sin\theta)^{4m}}, \tag{3.10c}$$

$$\omega = -\frac{2\beta(\rho/\beta)^{2m+2}(1-\cos\theta)^{2m+1}}{(\rho/\beta)^{4m+2} - (\sin\theta)^{4m+2}}. \tag{3.10d}$$

This metric is not strictly speaking new as it becomes the metric of Kinnersley and Kelley (1974) when we apply the NUT transformation, $\xi' = -i\xi$. When $m = 0$, it is an (extreme Kerr)-NUT metric derived by attaching the NUT parameter, $-\frac{1}{2}\pi$, to an extreme Kerr metric of mass β. If the method of 'distinguished limits' of Kinnersley and Kelley is applied to the $h \neq 0$ generalisations of the TS metrics with $\delta \geqslant 2$, no further solutions outside the class (3.10) result.

3.2. *Contractions arising from* (3.3) *and* (3.4)

The co-ordinate contractions, (3.3*b*) and (3.4*b*), give rise to an interesting family of solutions recently published by Ernst (1977). We shall describe several interesting properties of these solutions and their relationship to the generalised TS solutions in a separate paper. Here, we shall be content with presenting their derivation and some new examples which take elementary functional forms.

Applying the limit (3.3*b*) to equation (2.9), we find

$$e^{2\gamma} = (\text{constant})\rho^{-2h}(1+\eta)^{-\delta^2}\bar{\Gamma}(\eta) \tag{3.11}$$

where $\eta = (1+\cos\theta)/(1-\cos\theta)$ is the limiting form of the co-ordinate η in (2.9). We shall use this new η rather than θ for convenience. Since δ and h do not require rescaling, the functions, $\bar{\Gamma}(\eta)$, $\Gamma(\eta)$, $H_4(\eta)$, $H_2(\eta)$, etc, satisfy the same differential equations and relationships as before. Applying the limit (3.4*b*) to (2.9), we find

$$e^{2\gamma} = (\text{constant})\,\rho^{2h}(1+1/\eta)^{-\delta^2}\bar{\Gamma}(1/\eta).$$

Since $\bar{\Gamma}(\eta)$ and $\eta^{\delta^2}\bar{\Gamma}(1/\eta)$ satisfy the same DE, with same h and δ, then

$$e^{2\gamma} = (\text{constant})\rho^{2h}(1+\eta)^{-\delta^2}\bar{\Gamma}(\eta) \tag{3.12}$$

also yields a class of solutions of Einstein's equations. When $h \neq 0$, (3.11) and (3.12) yield quite distinct classes of solutions for a particular $\bar{\Gamma}(\eta)$. Combine the two cases into a single formula by replacing h in (3.11) by $\epsilon_1 h$ where $\epsilon_1 = \pm 1$.

To construct the full metric, first calculate, from (A.6a, b, c),

$$A_{[\rho,\theta]} = \rho^{-2}(H_2^2 + \delta^2 - 2\epsilon_1 h), \tag{3.13a}$$

$$B_{[\rho,\theta]} = -\rho^{-1}\eta^{-1/2}(H_4 + \epsilon_1 h(\eta - 1)), \tag{3.13b}$$

$$C_{[\rho,\theta]} = \eta(\sigma_1^2 + \sigma_2^2) + 2\epsilon_1 h, \tag{3.13c}$$

$$D_{[\rho,\theta]} = \rho^{-1}\eta^{-1/2}(1+\eta)H_1. \tag{3.13d}$$

Then the F equation I(3.13) takes the very simple form,

$$\rho^2 F_{\rho\rho\rho} + 3\rho F_{\rho\rho} + (1 - \delta^2 + 2\epsilon_1 h)F_\rho = 0,$$

and has linearly independent solutions,

$$F = \rho^k, \qquad k = 0, \quad \pm(\delta^2 - 2\epsilon_1 h)^{1/2},$$

$\delta^2 \neq 2\epsilon_1 h$. Thus, reparametrising according to (2.11), observing that $(\delta^2 - 2\epsilon_1 h)^{1/2} = n + m - \epsilon_1 m$, we see that the Ernst potential must take the two forms,

$$\mathscr{E} = e^{2u} + i\psi = (R(\eta) + iS(\eta))\rho^{n+m-\epsilon_1 m}, \tag{3.14a}$$

$$\mathscr{E}' = e^{2u'} + i\psi' = \mathscr{E}^{-1} = \frac{R - iS}{R^2 + S^2}\rho^{-(n+m-\epsilon_1 m)}, \tag{3.14b}$$

for two particular values of the NUT parameter (exceptional case $\delta^2 = 2\epsilon_1 h$: see (3.20) below). Comparing (3.14a) and (3.13a, b, c, d) with I(2.6a, b, c, d), we find the relations,

$$(n + m - \epsilon_1 m)S/R = H_2, \tag{3.15a}$$

$$[H_2^2 + (n + m - \epsilon_1 m)^2]R_\eta/R$$
$$= (n + m - \epsilon_1 m)\eta^{-1}(1+\eta)^{-1}[H_4 + \epsilon_1 m(n+m)(\eta - 1)] + 2\eta^{-1}H_1 H_2. \tag{3.15b}$$

Let us write down simple closed-form solutions for the cases, $n = 1$ and $n = 2$. The planned paper mentioned at the beginning of this subsection will provide recurrence formulae for $n \to n \pm 1$ and $m \to m \pm \frac{1}{2}$. If $n = 1$, we have

$$\mathscr{E}(\epsilon_1 = +1) = \rho\left[\frac{p^2\eta^{2m+1} - q^2}{\eta^m(1+\eta)} + ipq(2m+1)\right], \tag{3.16}$$

$$\mathscr{E}(\epsilon_1 = -1) = \rho^{2m+1}\frac{\eta^m(p\eta^{m+1} + iq)}{(1+\eta)^{2m+1}(p\eta^m - iq)}, \tag{3.17}$$

$p^2+q^2=1$. If $n=2$, we have

$$\mathscr{E}(\epsilon_1=+1)=\rho^2\bigg[\frac{(p^2\eta^{2m+2}+q^2)^2-4p^2q^2(m+1)^2(1+\eta)^2\eta^{2m+1}}{\eta^m(1+\eta)^2(p^2\eta^{2m+2}+q^2)} + \frac{ipq(m+1)\{p^2\eta^{2m+2}[2m+3+(2m+1)\eta]-q^2[2m+1+(2m+3)\eta]\}}{(1+\eta)(p^2\eta^{2m+2}+q^2)}\bigg], \tag{3.18}$$

$$\mathscr{E}(\epsilon_1=-1)=\rho^{2m+2}\frac{\eta^m[p^2\eta^{2m+3}-q^2+ipq(2m+3)(1+\eta)\eta^{m+1}]}{(1+\eta)^{2m+2}[p^2\eta^{2m+1}-q^2-ipq(2m+1)(1+\eta)\eta^m]}. \tag{3.19}$$

Solutions (3.17), (3.18), (3.19) and similar solutions for higher values of n are all new when $m\neq 0$. When $m=0$, they reduce to the series of elementary solutions of Ernst (1977). The solution (3.16) is a Lewis solution which, when $m=0$, is actually a flat space–time in unusual co-ordinates. If $n=0$, the solutions of this type are either static or are in the Papapetrou–Ehlers class.

It is interesting that the Newman–Penrose components, Ψ_0, Ψ_2 and Ψ_4, of the Weyl tensor (Newman and Penrose 1962) are analytic on both branches, $\eta=0$ and $\eta=\infty$, of the symmetry axis and are asymptotically vanishing in the cases, $\epsilon_1=-1$, $m=1$, $n=1,2,3,\ldots$, and for the metrics derived from $\mathscr{E}'=\mathscr{E}^{-1}$ when $m=0$, $n=1,2,3,\ldots$. Thus it may be instructive to explore the maximal analytic extensions of these metrics although either Ψ_0 or Ψ_4 is always singular at zeros of $\bar{\Gamma}(\eta)$.

The exceptional cases where the forms (3.14*a*, *b*) for $\mathscr{E}$ are not appropriate are the two equivalent cases, $\epsilon_1=+1$, $n=0$, $\delta^2=2m^2$, $h=m^2$ and $\epsilon_1=-1$, $n=-2m$, $\delta^2=2m^2$, $h=-m^2$. Now, for a particular NUT parameter, $\mathscr{E}$ must take the form,

$$\mathscr{E}=H_2^{-1}+i(S(\eta)+\ln\rho), \tag{3.20}$$

where

$$S_\eta=\eta^{-1}(1+\eta)^{-1}H_2^{-2}[H_4+m^2(\eta-1)]. \tag{3.21}$$

An infinite sequence of elementary solutions in this class may be obtained from the $\epsilon_1=-1$ cases, $n=1,2,3,\ldots$, by letting $m\to-\frac{1}{2}n$ whilst adjusting p^2, q^2 and suitable values of α_1, α_2, α_3, α_4 in I(1.11) in such a way that $p^2\eta^{n+2m}+(-1)^nq^2$ approaches $a+b\ln\eta$ with b/a finite. A simple example, where $h=\frac{1}{4}$, $\delta=1/\sqrt{2}$, is

$$\mathscr{E}=\frac{\ln\eta+a}{\eta^{1/2}+i}-i\ln(1+1/\eta)+i\ln\rho, \tag{3.22a}$$

$$e^{2\gamma}=\rho^{-1/2}\eta^{1/4}(1+\eta)^{-1/2}(\ln\eta+a). \tag{3.22b}$$

3.3. *Contractions arising from* (3.5)

The third contraction (3.5*b*) leads to separable solutions of the form,

$$\gamma=2h_1z+\int r^{-1}k(r)\,dr, \tag{3.23}$$

where $k(r)$ satisfies the differential equation,

$$(rk''-k')^2=8(rk'^2-2kk'-4h_1^2r)(-k'-2\delta_1^2r), \tag{3.24}$$

the prime denoting d/dr, and h_1 and δ_1 are constants. These solutions appear to be quite new, except for a few special cases known to Lewis (1932), even though, like the previous family, they may be obtained as separable solutions of the Ernst $\mathscr{E}$ equation.

Define

$$\tilde{H}_2(r)=(-\tfrac{1}{2}r^{-1}k'-\delta_1^2)^{1/2},$$
$$\tilde{H}_1(r)=\tfrac{1}{4}(k'^2-2r^{-1}kk'-4h_1^2)^{1/2},$$

with signs chosen so that $\tilde{H}_2'=-4r^{-1}\tilde{H}_1$. The correspondence with the generalised TS solutions is given by setting

$$H_4(\eta)=k(r),\qquad H_2(\eta)=\kappa\tilde{H}_2(r),\qquad H_1(\eta)=\kappa\tilde{H}_1(r),\qquad h=\kappa h_1,\qquad \delta=\kappa\delta_1$$

before letting $\kappa\to\infty$. Now, from I(2.2),

$$A_{[z,r]}=-2r^{-1}k',\qquad B_{[z,r]}=4h_1r^{-1},$$
$$C_{[z,r]}=-2r^{-1}k'+4r^{-2}k,\qquad D_{[z,r]}=4r^{-1}\tilde{H}_1.$$

Hence, the F equation I(3.13) takes the very simple form,

$$F_{zzz}-4\delta_1^2F_z=0,$$

from which we deduce, for a particular NUT parameter, the Ernst potential,

$$\mathscr{E}=(-\tfrac{1}{2}r^{-1}k')^{-1/2}(\delta_1-\mathrm{i}\tilde{H}_2)\exp\left(-4\delta_1h_1\int k'^{-1}\,\mathrm{d}r+2\delta_1z\right),\tag{3.25}$$

when $\delta_1\neq0$. When $\delta_1=0$, the Ernst potential takes the form,

$$\mathscr{E}=\tilde{H}_2^{-1}+\mathrm{i}\left(4h_1\int k'^{-1}\,\mathrm{d}r-2z\right),\tag{3.26}$$

unless $\tilde{H}_2=0$ in which case the metric is a cylindrically symmetric Weyl solution.

Now the DE (3.24) transforms into (3.7) under the change of variables,

$$r=\lambda^{-1/2},\qquad k(r)=-\lambda^{-1}\tilde{k}(\lambda),$$

so the exact solution (3.10a) leads immediately to the solution,

$$\mathrm{e}^{2\gamma}=(\text{constant})[(\beta r)^{2(m+1)^2}-(\beta r)^{2m^2}].$$

However, this solution is not new but is precisely the general rotating cylindrically symmetric solution of Lewis (1932). It is a special case of (3.26), which was also known to Lewis (1932, see the solution on pages 184–5 and compare it with $\mathrm{e}^{2u}=\tilde{H}_2^{-1}$, $\omega=-k(r)-2h_1z$).

4. Vacuum metrics with two Killing vectors and a second-rank Killing tensor

In this section, we shall derive all vacuum metrics of the form,

$$\mathrm{d}s^2=g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j=g_{44}\,\mathrm{d}t^2+2g_{34}\,\mathrm{d}\phi\,\mathrm{d}t+g_{33}\,\mathrm{d}\phi^2+g_{11}\,\mathrm{d}r^2+g_{22}\,\mathrm{d}z^2,\tag{4.1}$$

$(x^1,x^2,x^3,x^4)=(r,z,\phi,t)$, $g_{ij}=g_{ij}(r,z)$, $g_{11}=g_{22}$, which possess a second-rank Killing tensor (Walker and Penrose 1970, Woodhouse 1975) whose components are functions of the non-ignorable co-ordinates, r and z, only. The form (4.1) presupposes the pair of commuting Killing vectors, $\partial/\partial\phi$ and $\partial/\partial t$, as well as the discrete reflection symmetry,

$(\phi, t) \to (-\phi, -t)$. Lewis (1932) showed that $\nabla^2(-\Delta)^{1/2} = 0$, where

$$\Delta \equiv g_{33}g_{44} - (g_{34})^2, \qquad \nabla^2 \equiv \partial^2/\partial r^2 + \partial^2/\partial z^2,$$

and so derived the canonical form (1.1) by setting $\Delta = -r^2$. However, this canonical choice of co-ordinates is not available if Δ is constant, in which case a suitable canonical metric form is (4.32) below, or if Δ is any function of $r + iz$ or $r - iz$ only, but in this case the metric cannot, in general, if its Ricci tensor vanishes, adopt the correct Lorentz signature of -2 (see (4.43) below).

Our approach differs substantially from other authors (with the exception of Hauser and Malhiot (1974) for the case of spherical symmetry) in that we make no assumptions regarding separability of the Hamilton–Jacobi (HJ) or Schrödinger equations or about the algebraic type of the Riemann tensor but solve Killing's equations coupled with Einstein's equations directly from first principles. Nevertheless, in the case of Lewis' metric form (1.1), we are led to essentially the same canonical form for metrics (vacuum or otherwise) possessing a Killing tensor as Carter (1968, equation (1)) who assumed that the HJ and Schrödinger equations are solvable by separation of variables (Carter was later aware that the assumption of Schrödinger separability can be avoided as it is a consequence of HJ separability—see Benenti (1976), Collinson and Fugere (1977)). But here, to pick out the vacuum metrics which adopt Carter's canonical form, we shall employ the γ equation I(2.7) which, by the duality principle explained in I, is satisfied by $\gamma - u + \frac{1}{4}\ln r$. Then four functional forms for $e^{2\gamma-2u}$ to be substituted are given by (1.3) above. Later, we notice a class of HJ and Schrödinger separable metrics of the form (4.32) below which do not appear on Carter's list (see his equations (4)–(19) with $\Lambda = e = 0$ classifying the solutions into types $[A]$, $[\tilde{B}(+)]$, $[\tilde{B}(-)]$ and $[D]$) as well as a number of metrics with non-Lorentz signature possessing a Killing tensor but which are not HJ separable.

Killing vectors $K^i(\partial/\partial x^i)$ and second-rank Killing tensors $K^{ij}(\partial/\partial x^i)(\partial/\partial x^j)$ may be defined as having the property that

$$K_i \, dx^i/ds = \text{constant}, \qquad K_{ij}(dx^i/ds)(dx^j/ds) = \text{constant}$$

are first integrals of the differential equations for a non-null geodesic. The defining equations for Killing vectors and tensors are, respectively,

$$K_{(i;j)} = 0, \qquad K_{(ij;k)} = 0, \tag{4.2a, b}$$

the semicolon denoting covariant differentiation. A Killing tensor $K^{ij}(\partial/\partial x^i)(\partial/\partial x^j)$ will be considered reducible (trivial, redundant) if K^{ij} is a linear combination with constant coefficients of g^{ij} and products of Killing vector components of the form K^iK^j and/or $K^{(i}L^{j)}$, and a set of irreducible Killing tensors will be considered distinct if no linear combination of them forms a reducible Killing tensor or zero. Clearly, the metric (4.1) always possesses the trivial Killing tensors,

$$(\partial/\partial\phi)^2, \qquad (\partial/\partial\phi)(\partial/\partial t), \qquad (\partial/\partial t)^2, \qquad g^{ij}(\partial/\partial x^i)(\partial/\partial x^j).$$

Further, individual metrics differing by a co-ordinate transformation will be considered equivalent and classes of metrics differing by a reparametrisation and/or co-ordinate transformation will be considered equivalent. In particular, the transformation group **L**, defined by I(1.5) or I(1.6), generates an equivalence class of metrics and, by theorem 2 of § 3 of I, we need provide only a single completed metric for a given $e^{2\gamma-2u}$. Also, for convenience, we shall regard two metrics as equivalent even if the co-ordinate transformation or reparametrisation involves the formal use of complex numbers. Thus, for example, the cylindrically symmetric static metric,

$$e^{2u} = ar^{2\lambda}, \qquad \omega = 0, \qquad e^{2\gamma-2u} = br^{2\lambda(\lambda-1)}, \tag{4.3}$$

a, b, λ constants, is equivalent under **L** with $(\beta_1, \beta_2, \beta_3, \beta_4) = (0, -i, -i, 0)$, the dual of $\xi' = -\xi$, to the same metric with λ replaced by $1-\lambda$. Similarly, by this convention, Carter's metrics of types $[\tilde{B}(+)]$ and $[\tilde{B}(-)]$ are equivalent to each other.

Now, the contravariant form of Killing's equations (4.2b) may be written

$$g^{l(k}K^{ij)}{}_{,l} = K^{m(k}g^{ij)}{}_{,m}, \tag{4.4}$$

the comma denoting partial differentiation. When $K^{ij} = K^{ij}(r, z)$, the 20 components of this tensor equation separate into two completely independent sets of 10 equations. Consider, first, the cases,

$$(ijk) = (333), (334), (344), (444), \tag{4.5a}$$

$$(ijk) = (113), (114), (123), (124), (223), (224), \tag{4.5b}$$

which involve only the tensor components, K^{13}, K^{14}, K^{23} and K^{24}. The four equations (4.5a) form a set of four linear homogeneous algebraic equations for K^{13}, K^{14}, K^{23} and K^{24}. Thus these components either all vanish or the coefficient determinant vanishes, i.e.,

$$(u_r\omega_z - u_z\omega_r)^2 - (1/r)\omega_z(u_r\omega_z - u_z\omega_r) + u_z^2\, e^{-4u} = 0,$$

using the metric form (1.1). Thus, from I(2,11), this may be written, $r^2J' = C'$, which is the partial DE,

$$r^2(\nabla^2\zeta)^2 = \zeta_r^2 + \zeta_z^2,$$

for $\zeta \equiv \gamma - u$, whose general integral is given by

$$\zeta_r = r(v_r^2 - v_z^2), \qquad \zeta_z = 2rv_rv_z, \qquad \nabla^2 v + (1/r)v_r = 0.$$

Hence $A' = 4v_r^2 + 1/r^2$, $B' = 4v_rv_z$, $C' = 4v_z^2$, $J' = (4/r^2)v_z^2$. The field equation I(1.1) simplifies to the single term, $-(64/r^4)v_z^4 = 0$. So

$$e^{2\zeta} \equiv e^{2\gamma-2u} = br^{2\lambda(\lambda-1)}, \tag{4.6}$$

b, λ constants. When $\lambda \neq \frac{1}{2}$, these metrics are the Lewis cylindrically symmetric metrics equivalent under **L** to the static metrics (4.3). The case $\lambda = \frac{1}{2}$ is the exceptional case in theorem 2 of § 3 of I and so (4.6) permits the entire family of Lewis solutions of the second kind defined by I(2.18), but in these cases the six equations (4.5b) assume very simple forms and permit only the cylindrically symmetric solution,

$$e^{2u} = r(c + d\ln r), \qquad \omega = \pm(c + d\ln r)^{-1}, \qquad e^{2\gamma-2u} = br^{-1/2}, \tag{4.7}$$

c, d, b constants. The only Killing tensors that result in either case are trivially reducible to the Killing vectors, $\partial/\partial z$, $\partial/\partial\phi$ and $\partial/\partial t$. Very similar conclusions apply to the metric forms (4.32), (4.34) and (4.43) below so that all non-trivial Killing tensors with $K^{ij} = K^{ij}(r, z)$ must have $K^{13} = K^{14} = K^{23} = K^{24} = 0$.

The remaining ten of Killing's equations (4.4) will now be written out in general orthogonal co-ordinates $(x^1, x^2, x^3, x^4) = (\rho, \tau, \phi, t)$, as in § 2 and § 3 of I, the orthogonality condition being $g^{12} = g^{21} = 0$. The cases, $(ijk) = (111)$, (112), (122) and (222), are, respectively,

$$g^{11}(K^{11})_\rho = (g^{11})_\rho K^{11} + (g^{11})_\tau K^{12}, \tag{4.8a}$$

$$g^{22}(K^{11})_\tau + 2g^{11}(K^{12})_\rho = (g^{11})_\rho K^{12} + (g^{11})_\tau K^{22}, \tag{4.8b}$$

$$g^{11}(K^{22})_\rho + 2g^{22}(K^{12})_\tau = (g^{22})_\rho K^{11} + (g^{22})_\tau K^{12}, \tag{4.8c}$$

$$g^{22}(K^{22})_\tau = (g^{22})_\rho K^{12} + (g^{22})_\tau K^{22}. \tag{4.8d}$$

These four equations would remain unaffected if we allowed the components K^{ij} to be functions of all four co-ordinates. The six cases, $(ijk) = (133), (233), (134), (234), (144)$ and (244), may be grouped into two sets of three equations as follows:

$$g^{11}(K^{lm})_\rho = (g^{lm})_\rho K^{11} + (g^{lm})_\tau K^{12}, \tag{4.9a}$$

$$g^{22}(K^{lm})_\tau = (g^{lm})_\rho K^{12} + (g^{lm})_\tau K^{22}, \tag{4.9b}$$

where $(lm) = (33), (34), (44)$.

Equations (4.8) will now be solved completely for the four unknown functions, K^{11}, K^{12}, K^{22} and $V \equiv e^{2\gamma-2u}$. Returning to the $(x^1, x^2) = (r, z)$ system, it is immediately obvious that $K^{11} - K^{22}$ and $2K^{12}$ are conjugate harmonic functions, i.e.,

$$K^{11} - K^{22} = R(z, r), \qquad 2K^{12} = Z(r, z), \qquad R_r = Z_z, \qquad R_z = -Z_r. \tag{4.10}$$

Now, equations (4.8) reduce to the pair,

$$(VK^{22})_z = -\tfrac{1}{2}V_r Z, \qquad (VK^{22})_r = -(VR)_r - \tfrac{1}{2}V_z Z. \tag{4.11a, b}$$

Eliminating K^{22}, we obtain the following linear partial DE for V:

$$Z(V_{rr} - V_{zz}) - 2RV_{rz} + 3Z_r V_r - 3Z_z V_z - 2Z_{zz}V = 0. \tag{4.12}$$

Although equations (4.9) may be used to restrict R and Z to very simple functions, it is instructive, first, to solve (4.12) for general R and Z. Define new conjugate harmonic functions, $X(r, z)$ and $Y(r, z)$, by

$$X_r = Y_z = (R^2 + Z^2)^{-1/2}[(R^2 + Z^2)^{1/2} + R]^{1/2}, \tag{4.13a}$$

$$X_z = -Yr = (R^2 + Z^2)^{-1/2}[(R^2 + Z^2)^{1/2} - R]^{1/2}, \tag{4.13b}$$

with signs chosen so that $X_r X_z = -Y_r Y_z = Z(R^2 + Z^2)^{-1}$. The general integral of (4.12) is

$$V = \frac{F(X) + G(Y)}{(R^2 + Z^2)^{1/2}}, \tag{4.14}$$

where F and G are arbitrary functions, provided $R^2 + Z^2 \neq 0$. Now, with new co-ordinate identification, $(x^1, x^2, x^3, x^4) = (X, Y, \phi, t)$, we have

$$g_{11} = g_{22} = -\tfrac{1}{2}[F(X) + G(Y)], \qquad g^{11} = g^{22} = -2[F(X) + G(Y)]^{-1} \tag{4.15}$$

and $g_{12} = g^{12} = 0$. The solution of (4.8) is

$$K^{11} = \frac{2G(Y)}{F(X) + G(Y)}, \qquad K^{12} = 0, \qquad K^{22} = -\frac{2F(X)}{F(X) + G(Y)} \tag{4.16}$$

(K^{ij} may be replaced by $\lambda K^{ij} + \mu g^{ij}$, λ, μ constants). Now equations (4.9) simplify to

$$(K^{lm})_X = -G(Y)(g^{lm})_X, \qquad (K^{lm})_Y = F(X)(g^{lm})_Y,$$

and are readily solved for both the metric and Killing tensor components. The results are

$$g^{lm}=\frac{F_{lm}(X)+G_{lm}(Y)}{F(X)+G(Y)},\tag{4.17}$$

$$K^{lm}=\frac{F(X)G_{lm}(Y)-G(Y)F_{lm}(X)}{F(X)+G(Y)}+k_{lm},\tag{4.18}$$

where $F_{lm}(X)$ and $G_{lm}(Y)$, $(lm)=(33), (34), (44)$, are arbitrary functions and k_{lm} are arbitrary constants. Note that $(R^2+Z^2)^{1/2}$ must also take the form, $f(X)+g(Y)$.

Comparing (4.15) and (4.17) with equation (1) of Carter (1968), we see that our form of the metric is identical (apart from notational differences) with that derived by Carter from the assumption of the separability of the HJ and Schrödinger equations. Thus all metrics of the form (4.1) with a Killing tensor, $K^{ij}=K^{ij}(r, z)$, have separable HJ and Schrödinger equations (provided $R^2+Z^2\neq 0$) and the variables of separation are X and Y. The next step, namely substitution of this general metric form into Einstein's equations, may be carried out in two ways. Translating into our notations, Carter's method essentially was to substitute equation (4.17) into Einstein's equations I(1.3*a*, *b*) and solve for the eight unknown functions F, F_{lm}, G, G_{lm}, a quite difficult task (in fact, Carter also included a Λ-term and charge parameter e; here $\Lambda=e=0$). But it is somewhat easier to substitute (4.14) into the dual of the γ equation I(2.7). It is then a very simple matter to construct the full metric and its Killing tensor. But, first, there is one more trick which considerably simplifies the analysis.

With $(\rho, \tau)=(r, z)$, eliminate K^{lm} from (4.9*a*, *b*) by cross-differentiation and use (4.10), (4.11) and (4.12) to obtain

$$Z[(Vg^{lm})_{rr}-(Vg^{lm})_{zz}]-2R(Vg^{lm})_{rz}+3Z_r(Vg^{lm})_r-3Z_z(Vg^{lm})_z-2Z_{zz}Vg^{lm}=0.\tag{4.19}$$

Taking the metric form (1.1) where $\Delta=-r^2$, multiply (4.19) throughout by g_{lm} and sum over (lm), counting the $(lm)=(34)$ case twice. We discover that u and ω can be eliminated completely using I(2.11*a*, *b*, *c*). The resulting equation is, remarkably,

$$6r^{-2}V(rZ_r-Z)=0\tag{4.20}$$

and so

$$R=\tfrac{1}{2}a_1(r^2-z^2)+b_1z+c_1, \qquad Z=a_1rz-b_1r.\tag{4.21}$$

Using the freedom to replace z by $z-z_0$, there are four canonical choices for the parameters, a_1, b_1 and c_1, yielding four co-ordinate systems (X, Y) simply related to the systems, (x, y), (ρ, θ), (σ, τ) and (r, z), defined in the appendix. These are:

Case 1: $a_1=1$, $b_1=0$, $c_1-\tfrac{1}{2}\kappa^2\neq 0$,

$$X=2\cosh^{-1}x,\quad Y=2\sin^{-1}y,\quad (R^2+Z^2)^{1/2}=\tfrac{1}{2}\kappa^2(x^2-y^2);\tag{4.22a}$$

Case 2: $a_1=1$, $b_1=0$, $c_1=0$,

$$X=2\ln\rho,\quad Y=-2\theta,\quad (R^2+Z^2)^{1/2}=\tfrac{1}{2}\rho^2;\tag{4.22b}$$

Case 3: $a_1=0$, $b_1=1$, $c_1=0$,

$$X=2\tau^{1/2},\quad Y=2\sigma^{1/2},\quad (R^2+Z^2)^{1/2}=\tfrac{1}{2}(\sigma+\tau);\tag{4.22c}$$

Case 4: $a_1=0, \quad b_1=0, \quad c_1=1,$

$$X=\sqrt{2}r, \quad Y=\sqrt{2}z, \quad (R^2+Z^2)^{1/2}=1. \tag{4.22d}$$

To solve Einstein's equations for case 1, substitute

$$V=e^{2\gamma-2u}=\frac{f(x)+g(y)}{x^2-y^2} \tag{4.23}$$

into the dual of the γ equation, where, from (4.14) and (4.22*a*), $f(x)=(2/\kappa^2)F(X)$, $g(y)=(2/\kappa^2)G(Y)$. While easier than Carter's method, it is still a tedious calculation, so the details will be omitted. Three classes of solutions emerge:

$$f(x)=a(x^2+1)+bx, \qquad g(y)=c(y^2+1)+dy, \qquad cb^2+ad^2=4ac(a+c); \tag{4.24}$$

$$f(x)=ax^2+bx+\tfrac{1}{2}kx\ \ln\frac{x-1}{x+1}, \qquad g(y)=-ay^2+dy+\tfrac{1}{2}ky\ \ln\frac{1+y}{1-y}; \tag{4.25}$$

$$f(x)=a(x^2-1)^{-1}, \qquad g(y)=a(1-y^2)^{-1}. \tag{4.26}$$

Solution (4.24) yields the Kerr–NUT, Kerr, Taub–NUT and Schwarzschild metrics and a limiting case, $a=c=0$, which can be a real Lorentz metric only if $d=\mp b$, (see (4.27) below). Solution (4.25) yields PE solutions of the second kind (see § 2 and corollary 1 in § 3 of I) and so normally have no application in general relativity unless they are equivalent under a complex element of **L** to a real Lorentz metric. The case, $a=k=0$, $d=\mp b$, common to (4.24), is such an example. Other cases may be interpreted as positive definite Riemannian metrics by setting $\phi=i\phi'$, $e^{2\gamma}<0$. Solution (4.26) yields the unphysical metric (4.3) with $\lambda=\frac{1}{2}\pm\frac{1}{2}i\sqrt{3}$ which turns up also in cases 2, 3 and 4. Of course, the solutions just named are particular members of an equivalence class generated by the group **L**.

The Kerr–NUT solution is normally given by Ernst's formula, $\xi=e^{i\lambda}(px-iqy)$, $p^2+q^2=1$. With this parametrisation, the constants, a, b, c, d, in (4.24) are given by $a=1$, $b=2p^{-1}\cos\lambda$, $c=q^2p^{-2}$, $d=2qp^{-2}\sin\lambda$, unless $p=0$, in which case $a=b=0$, $c=1$, $d=2\sin\lambda$. The limiting case, $a=c=0$, $d=\mp b$, is equivalent to the Weyl metric

$$e^{2u}=\kappa(x\pm1)(1+y), \qquad \omega=0, \tag{4.27}$$

which is a flat space–time which turns up also in cases 2 and 3 if we replace $z\pm\kappa$ by z. Since it is quite straightforward to construct the full metric and Killing tensors from equations (4.15)–(4.18), we shall not write out the results explicitly. However, note that not all of the solutions given by (4.25) and (4.26) have Killing tensors as (4.17) sometimes provides a further constraint. In fact, (4.26) does not survive at all (except in case 4 where the trivial Killing tensor $(\partial/\partial z)^2$ turns up) and in (4.25), we must have $k=0$.

The proper Kerr–NUT and Kerr metrics ($p\neq0$, $q\neq0$) are contained in Carter's type [A] and their Killing tensors are irreducible. The Taub–NUT and Schwarzschild metrics ($q=0$ or, by $(x,y)\to(y,x)$ symmetry, $p=0$), having four Killing vectors, are in type [$\tilde{B}(\pm)$] and their Killing tensors are reducible. The Killing tensor derived from (4.25) with $k=0$ is irreducible except for the three cases, a = b = 0 and $a=d=0$, having four Killing vectors, and $a=0$, $d=\mp b$, having ten (since it is flat).

To solve Einstein's equations for cases 2, 3 and 4, substitute

$$V=\frac{f(\rho)+g(\theta)}{\rho^2}, \qquad V=\frac{f(\sigma)+g(\tau)}{\sigma+\tau}, \qquad V=f(r)+g(z) \tag{4.28a, b, c}$$

into the dual of the γ equation, respectively. In each case, two classes of solutions correspond to contractions of (4.24) and (4.25) arise. The first class may be written:

$$f(\rho)=\rho^2+(2m\cos\lambda)\rho, \qquad g(\theta)=m^2(1+\cos^2\theta+2\sin\lambda\cos\theta); \tag{4.29a}$$

$$f(\sigma)=[(p\cos\tfrac{1}{2}\lambda)\sigma+q\sin\tfrac{1}{2}\lambda]^2, \qquad g(\tau)=[(q\cos\tfrac{1}{2}\lambda)\tau+p\sin\tfrac{1}{2}\lambda]^2; \tag{4.29b}$$

$$f(r)=(r^2-a^2)^2, \qquad g(z)=(2az+b)^2. \tag{4.29c}$$

Solution (4.29a) is the (extreme Kerr)–NUT metric derived from $\xi=e^{i\lambda}(\rho/m-i\cos\lambda)$, $m\neq 0$. It is Carter type $[A]$ and its Killing tensor is irreducible. Solution (4.29b) corresponds to

$$\xi=e^{i\lambda}\frac{p(\sigma+1)+iq(\tau-1)}{p(\sigma-1)+iq(\tau+1)}.$$

The Killing tensor is irreducible when $p\neq 0$ and $q\neq 0$ (type $[A]$). When $q=0$ (equivalent to $p=0$ under $(\sigma,\tau)\to(\tau,\sigma)$), the solution is in the PE class (type $[\tilde{B}(\pm)]$, K^{ij} reducible, four Killing vectors) and contains as special cases the plane symmetric space–time ($\lambda=0$) and a flat space–time ($\lambda=\pi$, type $[D]$, equivalent to (4.27)). Solution (4.29c) may be generated from Minkowski space–time in cylindrical co-ordinates by applying an element of **L**, thence an element of **P** (and, by equivalence, a further **L**) and is type $[A]$ with irreducible Killing tensor when $a\neq 0$. When $a=0$, (4.29c) becomes the PE solution, $\xi=e^{i\lambda}(1+r^2)/(1-r^2)$, $b=\cot\frac{1}{2}\lambda$ (type $[\tilde{B}(\pm)]$, K^{ij} reducible, four Killing vectors). The chosen normalisation omits Minkowski space–time itself (type $[D]$).

The PE solutions of the second kind satisfying (4.28a, b, c) are given by

$$f(\rho)=a\rho^2+b\rho-k, \qquad g(\theta=c\cos\theta+k\cos\theta\ln\cot\tfrac{1}{2}\theta; \tag{4.30a}$$

$$f(\sigma)=a\sigma^2+b\sigma+c+k\ln\sigma, \qquad g(\tau)=-a\tau^2+b\tau-k\ln\tau; \tag{4.30b}$$

$$f(r)=ar^2+c+k\ln r, \qquad g(z)=-2az^2+bz. \tag{4.30c}$$

As in the case of (4.25), when we compare the full metrics with (4.17), we find that $k=0$ in (4.30a, b, c). The Killing tensors derived from each of (4.30a, b, c) with $k=0$ will be irreducible whenever neither of the functions f and g is constant (or zero). When either f or g is constant, the space either has four Killing vectors or is flat. The special case, $V=a+b/\rho=a+2b/(\sigma+\tau)$, is common to case 2 and case 3 and, indeed, to case 1 if we replace z by $z\pm\kappa$ so that V becomes $a+\kappa^{-1}b/(x\pm y)$. When $a\neq 0$, $b\neq 0$, this space has four Killing vectors and one irreducible Killing tensor from case 3, the case 2 tensor being reducible and the case 1 tensor being a linear combination of the case 2 and case 3 tensors. The further special case, $V=b/\rho$, is flat and is equivalent under a complex element of **L** to the Carter type $[D]$ metric, (4.29b) with $q=0$, $\lambda=\pi$, mentioned above. In a similar way, $V=bz+c$ has an irreducible case 3 Killing tensor and a reducible case 4 tensor.

Several other classes of solutions of the form (4.28a, b, c) satisfy the dual of the γ equation. When $f(\rho)=0$, case 2 allows all the Ernst solutions given by (3.11) and (3.14a) or (3.20) with $\epsilon_1=+1$, $n=2-2h$, $\delta^2=4-6h+4h^2$ including the class generated by **P** from $\mathscr{E}=(1-\cos\theta)/(1+\cos\theta)$, $e^{2\gamma}=\rho^{-2}\sin^2\theta$. Substitution of these solutions into (4.17) is quite straightforward and we find that the only surviving solutions, apart from (4.30a) with $a=b=k=0$, are the PE solutions, $\xi=e^{i\lambda}\sec\theta$, $g(\theta)=1+\cos^2\theta+2\cos\lambda\cos\theta$ (type $[\tilde{B}(\pm)]$, K^{ij} reducible, four Killing vectors). Similarly, when $g(z)=0$, *Case 4* allows all the solutions given by (3.23) and (3.25) with

$\delta_1 = 2h_1$, (3.26) with $h_1 = 0$, the cylindrically symmetric Weyl and Lewis solutions and the class generated by **P** from the flat space-time, $\mathscr{E} = r^2$, $e^{2\gamma} = r^2$. But, again from (4.17), the only surviving solutions are (4.29*c*) with $a = 0$ already described and the cylindrically symmetric solutions whose only Killing tensor is the trivial tensor $(\partial/\partial z)^2$. In addition, (4.28*c*) allows the entire family of Lewis solutions of the second kind, I(2.18), but (4.17) restricts these to the single solution (4.7) with trivial Killing tensor $(\partial/\partial z)^2$. The one remaining solution of the dual of the γ equation not yet mentioned is the isolated case 3 solution, $f(\sigma) = \sigma^{-1}$, $g(\rho) = \tau^{-1}$, equivalent to (4.26) and appearing in all four cases, but admitting only the trivial case 4 tensor $(\partial/\partial z)^2$.

This completes the discussion of metrics in the Lewis canonical form (1.1) and we no longer make use of the theory of I. The conclusion reached so far is that all vacuum metrics of the form (1.1) with a Killing tensor, $K^{ij} = K^{ij}(r, z)$, have separable HJ and Schrödinger equations and all those with Lorentz signature are listed by Carter (1968) and are algebraically special, being Petrov type D or flat. When we extend the argument to metrics of the form (4.1) not reducible to Lewis' canonical form, some new and interesting results emerge: four classes of metrics with separable HJ and Schrödinger equations and Lorentz signature outside Carter's list, four classes of metrics with signature zero where an irreducible Killing tensor cannot be obtained by separation of variables in the HJ equation and a class of Lorentz metrics with a Killing tensor depending explicitly on the ignorable co-ordinates, ϕ and t.

The argument of Lewis (1932) for choosing $\Delta = -r^2$ in (4.1) breaks down if Δ is constant or a function of $\alpha = r + iz$ or $\beta = r - iz$ only. Thus, we may make two other choices, $\Delta = -1$ and $\Delta = -\alpha^2$. In the latter case, and sometimes in the former, we shall take α and β as new real variables (i.e. r real, z pure imaginary) so that the metric is real valued with signature zero. Now Einstein's vacuum equations for the metric,

$$ds^2 = e^{2u}(dt - \omega\, d\phi)^2 - e^{-2u}\, d\phi^2 - e^{2\gamma - 2u}(dr^2 + dz^2), \tag{4.31}$$

where $\Delta = -1$, are very easily solved and factorise into two distinct cases. In one case, one equation reads, $\nabla^2(\gamma - u) = 0$, $\nabla^2 = \partial^2/\partial r^2 + \partial^2/\partial z^2 = 4\partial^2/\partial\alpha\,\partial\beta$, so that we may choose canonical co-ordinates such that $e^{2\gamma - 2u} \equiv 1$. The remaining equations reduce to $\nabla^2(e^{2u}) = 0$ and $\omega = e^{-2u}$ after employing a co-ordinate transformation of the form, $\phi' = \pm\phi$, $t' = t + \lambda\phi$, λ constant. The metric now assumes the canonical form,

$$ds^2 = e^{2u}\, dt^2 - 2\, d\phi\, dt - (dr^2 + dz^2), \qquad \nabla^2(e^{2u}) = 0, \tag{4.32}$$

and is algebraically special, being Petrov type N or flat.

The other class of metrics of the form (4.31) with vanishing Einstein tensor (the word 'vacuum' being inappropriate as these metrics cannot adopt a Lorentz signature) may be derived from (4.32) by a discrete mapping analogous to the Neugebauer–Kramer mapping I(1.13). Defining ψ by

$$\psi_r = e^{4u}\omega_z = -(e^{2u})_z, \qquad \psi_z = -e^{4u}\omega_r = (e^{2u})_r,$$

the mapping is

$$e^{2u'} = e^{-2u}, \qquad \omega' = i\psi, \qquad e^{2\gamma'} = -e^{2\gamma - 2u}. \tag{4.33}$$

So, formally setting $\phi = i\phi'$, we obtain the positive definite Riemannian four-space,

$$ds^2 = e^{-2u}(dt + \psi\, d\phi')^2 + e^{2u}(d\phi'^2 + dr^2 + dz^2), \tag{4.34}$$

$$\nabla^2(e^{2u}) = 0, \qquad \psi_r = -(e^{2u})_z, \qquad \psi_z = (e^{2u})_r.$$

Alternatively, if α and β are taken as real variables, then the metrics (4.32) and (4.34) become the signature-zero metrics,

$$ds^2 = e^{2u}\,dt^2 - 2\,d\phi\,dt - d\alpha\,d\beta, \tag{4.35}$$

$$ds^2 = e^{-2u}(dt - \omega'\,d\phi)^2 + e^{2u}(-d\phi^2 + d\alpha\,d\beta), \tag{4.36}$$

respectively. In all four cases, the formal solution of Einstein's equations may be written,

$$e^{2u} = f_1(\alpha) + g_1(\beta), \qquad \omega' = f_1(\alpha) - g_1(\beta), \qquad \psi = -i\omega'. \tag{4.37}$$

Now, in the earlier analysis of Killing's equations, the paragraph containing equations (4.10)–(4.18) still applies here except that we must allow the possibility that $R^2 + Z^2 = 0$. Consider the metrics (4.32) and (4.35) first. The simplifying equation analogous to (4.20) here is $Z_{zz} = 0$, obtained from (4.12) with $V = 1$, so that

$$R = \tfrac{1}{2}a(r^2 - z^2) + br + cz + d, \qquad Z = arz - cr + bz + e. \tag{4.38}$$

Next, substituting $Vg^{33} = -(f_1(\alpha) + g_1(\beta))$ into (4.19), we find

$$[a\alpha^2 + 2(b - ic)\alpha + 2(d + ie)]f_1'' + [3a\alpha + 3(b - ic)]f_1' = k, \tag{4.39a}$$

$$[a\beta^2 + 2(b + ic)\beta + 2(d - ie)]g_1'' + [3a\beta + 3(b + ic)]g_1' = k, \tag{4.39b}$$

k constant.

Since the form of the metric and Einstein's equations are preserved by the Euclidean group of transformations, $r' = r\cos\delta - z\sin\delta + r_0$, $z' = r\sin\delta + z\cos\delta + z_0$, δ, r_0, z_0 constants, there are eight canonical choices for the parameters, a, b, c, d, e. The first four yield real Lorentz metrics of the form (4.32) while the last four yield signature-zero metrics of the form (4.35). They are:

Case 1: $a = 1$, $b = c = 0$, $d = \tfrac{1}{2}\kappa^2 \neq 0$, $e = 0$,

$$X = 2\cosh^{-1} x, \qquad Y = 2\sin^{-1} y, \qquad (R^2 + Z^2)^{1/2} = \tfrac{1}{2}\kappa^2(x^2 - y^2),$$

$$e^{2u} = (x^2 - y^2)^{-1}[kx(x^2 - 1)^{1/2}\cosh^{-1} x + \lambda x(x^2 - 1)^{1/2} + \nu x^2 - ky(1 - y^2)^{1/2}\sin^{-1} y + \mu y(1 - y^2)^{1/2} - \nu y^2]; \tag{4.40a}$$

Case 2: $a = 1$, $b = c = 0$, $d = e = 0$,

$$X = 2\ln\rho, \qquad Y = -2\theta, \qquad (R^2 + Z^2)^{1/2} = \tfrac{1}{2}\rho^2,$$

$$e^{2u} = \rho^{-2}[k\rho^2\ln\rho + \nu\rho^2 + \lambda\cos 2\theta + \mu\sin 2\theta]; \tag{4.40b}$$

Case 3: $a = 0$, $b = 0$, $c = 1$, $d = e = 0$,

$$X = 2\tau^{1/2}, \qquad Y = 2\sigma^{1/2}, \qquad (R^2 + Z^2)^{1/2} = \tfrac{1}{2}(\sigma + \tau),$$

$$e^{2u} = (\sigma + \tau)^{-1}[-\tfrac{1}{3}k\sigma^2 + \lambda\sigma^{1/2} + \nu\sigma + \tfrac{1}{3}k\tau^2 + \mu\tau^{1/2} + \nu\tau]; \tag{4.40c}$$

Case 4: $a = 0$, $b = c = 0$, $d = 1$, $e = 0$,

$$X = \sqrt{2}r, \qquad Y = \sqrt{2}z, \qquad (R^2 + Z^2)^{1/2} = 1,$$

$$e^{2u} = \tfrac{1}{2}k(r^2 - z^2) + \lambda r + \mu z + \nu; \tag{4.40d}$$

Case 5: $a=1,\quad b=c=0,\quad e=\mathrm{i}d,\quad d=\frac{1}{4}\kappa^2,$

$$X=\ln\bar{x},\qquad Y=-\mathrm{i}\ln\bar{y},\qquad (R^2+Z^2)^{1/2}=\tfrac{1}{4}\kappa^2(\bar{x}+\bar{y}),$$

where

$$\bar{x}=\alpha[\beta+(\beta^2+\kappa^2)^{1/2}],\qquad \bar{y}=\alpha[\beta+(\beta^2+\kappa^2)^{1/2}]^{-1},$$
$$e^{2u}=(\bar{x}+\bar{y})^{-1}[\tfrac{1}{2}k\bar{x}\ln\bar{x}+\lambda/\bar{x}+\mu\bar{x}+\tfrac{1}{2}k\bar{y}\ln\bar{y}+\lambda/\bar{y}+\nu\bar{y}]; \tag{4.40e}$$

Case 6: $a=0,\quad b=1,\quad c=-\mathrm{i},\quad e=-\mathrm{i}d,\quad d=\frac{1}{4}\bar{\kappa}^2\neq 0,$

$$X=\bar{\sigma},\qquad Y=-\mathrm{i}\bar{\tau},\qquad (R^2+Z^2)^{1/2}=\tfrac{1}{2}\bar{\kappa}(\bar{\sigma}-\bar{\tau})$$

where

$$\bar{\sigma}=\alpha/\bar{\kappa}+\beta^{1/2},\qquad \bar{\tau}=\alpha/\bar{\kappa}-\beta^{1/2},$$
$$e^{2u}=(\bar{\sigma}-\bar{\tau})^{-1}[\tfrac{1}{6}k\bar{\sigma}^3+\lambda\bar{\sigma}^2+\nu\bar{\sigma}-\tfrac{1}{6}k\bar{\tau}^3-\lambda\bar{\tau}^2-\nu\bar{\tau}+\mu]; \tag{4.40f}$$

Case 7: $a=0,\quad b=1,\quad c=-\mathrm{i},\quad d=e=0,\quad k=0,$

$$X,\ Y\ \text{not defined},\qquad R^2+Z^2=0,$$
$$e^{2u}=f_1(\alpha)+\mu\beta^{-1/2},\qquad f_1(\alpha)\ \text{arbitrary} \tag{4.40g}$$

Case 8: $a=0,\quad b=c=0,\quad d=1,\quad e=\mathrm{i},\quad k=0,$

$$X,\ Y\ \text{not defined},\qquad R^2+Z^2=0,$$
$$e^{2u}=f_1(\alpha)+\mu\beta,\qquad f_1(\alpha)\ \text{arbitrary}. \tag{4.40h}$$

A co-ordinate transformation of the form I(1.6) with $\beta_3=0$ may be used to set $\nu=0$ in cases 1–6.

Cases 1–4 have separable HJ and Schrödinger equations and Lorentz signature but lie outside Carter's list, except for the flat space-time, $e^{2u}=\lambda r+\mu z+\nu$ (case 4 with $k=0$, case 3 with $\lambda=\mu=0$), which is contained in type $[A]$ and Minkowski space–time, $e^{2u}=1$, contained in type $[D]$. The Killing tensor for case 1, written out in full, is

$$\begin{aligned}2(x^2-y^2)K^{ij}\Big(\frac{\partial}{\partial x^i}\Big)\Big(\frac{\partial}{\partial x^j}\Big)&\\ =-y^2(x^2-1)\Big(\frac{\partial}{\partial x}\Big)^2&-x^2(1-y^2)\Big(\frac{\partial}{\partial y}\Big)^2\\ &+\kappa^2[(1-y^2)(kx(x^2-1)^{1/2}\cosh^{-1}x+\lambda x(x^2-1)^{1/2}+\nu x^2)\\ &+(x^2-1)(ky(1-y^2)^{1/2}\sin^{-1}y-\mu y(1-y^2)^{1/2}+\nu y^2)]\Big(\frac{\partial}{\partial\phi}\Big)^2\end{aligned} \tag{4.41}$$

and is irreducible if not all of k, λ and μ are zero. The Killing tensor for case 7 is

$$K^{ij}\Big(\frac{\partial}{\partial x^i}\Big)\Big(\frac{\partial}{\partial x^j}\Big)=-\alpha\Big(\frac{\partial}{\partial\alpha}\Big)\Big(\frac{\partial}{\partial\beta}\Big)+\beta\Big(\frac{\partial}{\partial\beta}\Big)^2+\frac{1}{4}\Big[-\alpha f_1(\alpha)+\int f_1(\alpha)\,d\alpha-\mu\alpha\beta^{-1/2}\Big]\Big(\frac{\partial}{\partial\phi}\Big)^2, \tag{4.42}$$

irreducible if $\mu \neq 0$ and $f_1(\alpha)$ not constant. This Killing tensor cannot be derived by separation of variables in the HJ equation in the manner normally considered (exception: $f_1(\alpha) = \lambda(\alpha - \alpha_0)^{-1/2} + \nu$, equivalent to case 3 with $k = 0$). However, a complete integral of the HJ equation may be obtained by substituting a separated solution of the form, $S = kt - h\phi + S_1(\alpha) + \beta^{1/2} S_2(\alpha)$, and the constant of the motion, $K_{ij}(\mathrm{d}x^i/\mathrm{d}s)(\mathrm{d}x^j/\mathrm{d}s)$, arises from the integration of the DE for S_2. The Killing tensors for the other six cases will not be written out as they are easily derived from (4.16) and (4.18).

Apart from the flat metrics already mentioned, the Killing tensor is reducible in case 4 (six Killing vectors if $k \neq 0$) and case 8 (four Killing vectors if $\mu \neq 0$, at least five if $\mu = 0$). Case 3 with $k = 0$ is particularly interesting as it has two irreducible Killing tensors with $K^{ij} = K^{ij}(r, z)$ as well as a third irreducible Killing tensor depending explicitly on the ignorable co-ordinates (given below in § 5). The first two may be derived by substituting (4.40*c*) with $k = 0$ into (4.39*a*, *b*) and observing that both b and c remain undetermined or else by separating the HJ equation in variables, $X = q\sigma^{1/2} + p\tau^{1/2}$, $Y = p\sigma^{1/2} - q\tau^{1/2}$. Similarly, in case 6 with $k = 0$, e^{2u} does not depend on $\bar{\kappa}$, so two Killing tensors will be found but one is reducible. Also, in § 5, we point out that case 2 with $k = 0$, case 2 with $k \neq 0$ and $\lambda = \mu = 0$ and case 4 with $k \neq 0$ also have an extra irreducible Killing tensor depending on ϕ and t.

The metrics (4.34) and (4.36) are now very easily analysed. Since $Vg^{33} = -1$ or $+1$, (4.19) shows that $Z_{zz} = 0$ and so R and Z are given by (4.38). Since $V = -(f_1 + g_1)$, (4.12) leads to the same equations (4.39*a*, *b*) as before. But also $Vg^{34} = -\mathrm{i}(f_1 - g_1)$ or $f_1 - g_1$ and $Vg^{44} = -4f_1 g_1$, so (4.19) forces $k = 0$ in (4.39*a*, *b*) but has no other effect. So we are led to the same eight cases (4.40*a*–*h*) as before but with $k = 0$. The construction of the Killing tensors is straightforward, so we shall not pursue the matter.

The remaining metric form to be considered is the signature-zero metric,

$$\mathrm{d}s^2 = e^{2u}(\mathrm{d}t - \omega\,\mathrm{d}\phi)^2 - \alpha^2 e^{-2u}\,\mathrm{d}\phi^2 - e^{2\gamma - 2u}\,\mathrm{d}\alpha\,\mathrm{d}\beta, \tag{4.43}$$

which has $\Delta = -\alpha^2$. The general solution with vanishing Einstein tensor may be shown to be

$$e^{2u} = \frac{a(\alpha)M(\beta) + b(\alpha)N(\beta)}{c(\alpha)M(\beta) + d(\alpha)N(\beta)}, \qquad \omega = f(\alpha) + \alpha\, e^{-2u}, \tag{4.44a, b}$$

$$e^{2\gamma - 2u} = \alpha^{-1/4} \exp\left(\int f'(\alpha) g(\alpha)\,\mathrm{d}\alpha\right)[c(\alpha)M(\beta) + d(\alpha)N(\beta)], \tag{4.44c}$$

where

$$ad - bc = 1, \qquad a'd - ad' + bc' - b'c = 1/2\alpha,$$

$$cd' - c'd = f'(\alpha)/2\alpha, \qquad ab' - a'b = g(\alpha),$$

the prime denoting $\mathrm{d}/\mathrm{d}\alpha$. A single constraint, namely

$$\mu(NM' - MN') = N^3, \quad \mu \text{ constant} \neq 0, \tag{4.45}$$

the prime denoting $\mathrm{d}/\mathrm{d}\beta$, is sufficient to ensure that (4.43) possesses a second-rank Killing tensor. These metrics have $R^2 + Z^2 = 0$ like cases 7 and 8 above and so are not HJ separable in the usual sense but it is easy to obtain a complete integral of the HJ equation of the form, $S = kt - h\phi + S_1(\alpha) + MN^{-1}S_2(\alpha)$. By changing variable, $\beta = \beta(\bar{\beta})$, the ratio M/N may be normalised to any desired functional form, e.g. $M/N = \beta$, in which case (4.45) reduces to $M = \mu\beta$, $N = \mu$. There are two other classes of solutions where the HJ and Schrödinger equations are separable in the variables, $\alpha\beta=\rho^2$ and $\beta/\alpha=-e^{2i\theta}$. These are

$$e^{2u}=\lambda\alpha^{1/2}NM^{-1}+\mu\alpha^2+\nu\alpha, \qquad \omega=\alpha\, e^{-2u}, \qquad e^{2\gamma-2u}=\alpha^{-1/2}M \tag{4.46}$$

and

$$e^{2u}=N(\alpha^{-1/2}M+\lambda N)^{-1}, \qquad \omega=\alpha^{1/2}MN^{-1}, \qquad e^{2\gamma-2u}=\alpha^{-1/2}M+\lambda N, \tag{4.47}$$

where, in both cases,

$$M=d_1\beta^{-1/2}+d_2\beta^{-3/2}, \qquad N=d_3+d_4\beta^{-2},$$

λ, μ, ν, d_1, d_2, d_3, d_4 constants. The special case, $d_4d_1^2+d_3d_2^2=0$, of (4.47) may be obtained from Carter's type $[A]$ (type $[\tilde{B}(\pm)]$ if $d_2=d_4=0$) by suitable formal manipulations of parameters. This metric has four Killing vectors (ten if $d_1=d_2=0$) and the Killing tensor is reducible.

5. Conclusion

In § 2 and § 3, we found all solutions of the γ equation I(2.7) of the four forms,

$$\gamma=h\ln\frac{1+\nu}{1-\nu}+\gamma_2(\eta), \qquad 2\tilde{h}s+\gamma_2(\lambda),$$
$$-h\ln\rho+\gamma_2(\theta), \qquad 2h_1z+\gamma_2(r), \tag{5.1}$$

where the co-ordinates, (ν,η) and (ρ,θ), are defined in the appendix and (s,λ) is defined by (3.2b). The first case yielded the generalised TS solutions and we wrote down a new closed-form solution which generalised the Kerr solution to non-zero h. The starting point in the derivation was a minor modification of TS 'Rule (a)'. The second case yielded a new ($\tilde{h}\neq0$) family of solutions which contains the rotating Curzon solution ($\tilde{h}=0$), the Kinnersley–Kelley solution and the extreme Kerr solution. The third case yielded the Ernst (1977) family of solutions and we provided some new closed-form solutions and the fourth case yielded another class of new solutions.

A most promising problem for future research would be to use the new formulation of Einstein's equations in I to study separable solutions of the γ equation of the form,

$$\gamma=\gamma_1(X)+\gamma_2(Y), \tag{5.2}$$

where (X, Y), which we may call 'harmonic co-ordinates', satisfy $X_r=Y_z, X_z=-Y_r$. All co-ordinate systems used in this paper except (α,β) are of this type, as can be seen by setting $X=\frac{1}{2}\ln[(1+\nu)/(1-\nu)]$, $Y=\cot^{-1}(\eta^{1/2})$ for (ν,η) and $X=s$, $Y=\lambda^{-1/2}$ for (s,λ), the remaining cases being given by the equations immediately above each of (4.22a, b, c, d) and (4.40e, f). Rather than substituting (5.2) into the γ equation with $\gamma_1(X)$ and $\gamma_2(Y)$ both undetermined, it is better to aim for $\gamma_1(X)=0$ or a simple function so that $\gamma_2(Y)$ satisfies a single fourth-order differential equation. It may be necessary to allow γ to take the more general form, $\gamma=R(\gamma_1+\gamma_2)$, where $R=R(X, Y)$ is some simple function. Of course, many other types of separability may be considered.

In § 4, we sought all vacuum metrics of the form (4.1), including those which cannot assume the Lewis canonical form (1.1) and metrics with non-Lorentz signature, which possess a second-rank Killing tensor whose components do not depend on the ignorable co-ordinates, ϕ and t. The principal result was that all metrics with Lorentz signature have separable Hamilton–Jacobi and Schrödinger equations and assume the canonical form of Carter (1968, equation (1)) although the four metrics (4.32) with e^{2u} given by (4.40*a*, *b*, *c*, *d*) do not appear on Carter's list. One may have been tempted to expect this result from Woodhouse's (1975) theorem that any metric with three irreducible Killing tensors has separable HJ equation even though, quite clearly, the conditions of the theorem are not met. To put an end to such speculation, consider the following interesting example: the vacuum metric,

$$ds^2=[\rho^n(\lambda\cos n\theta+\mu\sin n\theta)+\nu]\,dt^2-2\,d\phi\,dt-(d\rho^2+\rho^2\,d\theta^2), \tag{5.3}$$

of the form (4.32), has the irreducible ($n\neq 0$ or 1) Killing tensor,

$$\begin{aligned}K^{ij}\Big(\frac{\partial}{\partial x^i}\Big)\Big(\frac{\partial}{\partial x^j}\Big)&=t\Big[\Big(\frac{\partial}{\partial\rho}\Big)^2+\frac{1}{\rho^2}\Big(\frac{\partial}{\partial\theta}\Big)^2\Big]-\rho\Big(\frac{\partial}{\partial\rho}\Big)\Big(\frac{\partial}{\partial\phi}\Big)\\&\quad+\Big[t\rho^n(\lambda\cos n\theta+\mu\sin n\theta)+\Big(\frac{1}{2}n+1\Big)(-\phi+\nu t)\Big]\Big(\frac{\partial}{\partial\phi}\Big)^2\\&\quad+\Big(\frac{1}{2}n+1\Big)t\Big(\frac{\partial}{\partial\phi}\Big)\Big(\frac{\partial}{\partial t}\Big),\end{aligned}\tag{5.4}$$

depending explicitly on the ignorable co-ordinates, ϕ and t. There appears to be no way of separating variables in the HJ equation or even of providing an elementary complete integral in any form for general n. When $n=-2$, the metric (5.3) becomes the case 2 metric (4.40*b*) with $k=0$ and so has another irreducible Killing tensor. When $n=-\frac{1}{2}$, (5.3) becomes the case 3 metric (4.40*c*) with $k=0$ and so has two more irreducible Killing tensors, as pointed out in § 4. A non-trivial limit as $n\to 0$ with irreducible Killing tensor takes the form, $e^{2u}=k\ln\rho+\mu_1\theta+\nu$, the case $\mu_1=0$ being contained in case 2. When $n=+2$, (5.3) becomes the case 4 metric (4.40*d*) with $k\neq 0$ after a rotation of axes. This metric has six Killing vectors but, nevertheless, the Killing tensor (5.4) is independent of them.

Acknowledgement

Special thanks are due to Ms D C Harriss for her careful typing of these two long and difficult manuscripts. I also wish to thank the referees for their helpful suggestions.

Appendix

We tabulate A, B and C in terms of γ in five selected co-ordinate bases. The co-ordinates, (ρ,θ), (x,y), (ν,η), (σ,τ) and (α,β) are defined by

$$r=\rho\sin\theta,\qquad z=\rho\cos\theta, \tag{A.1}$$

$$r=\kappa(x^2-1)^{1/2}(1-y^2)^{1/2},\qquad z=\kappa xy,\qquad \kappa\text{ constant}, \tag{A.2}$$

$$\nu=y/x,\qquad \eta=(x^2-1)/(1-y^2), \tag{A.3}$$

$$\sigma=\rho(1+\cos\theta),\qquad \tau=\rho(1-\cos\theta), \tag{A.4}$$

$$\alpha=r+iz,\qquad \beta=r-iz. \tag{A.5}$$

$A_{[r,z]}$, $B_{[r,z]}$ and $C_{[r,z]}$ are given by equations (2.2). From (2.9),

$$A_{[\rho,\theta]} = -2\gamma_{\rho\rho} - 2\rho^{-2}\gamma_{\theta\theta} - 2\rho^{-2}(\cot\theta)\gamma_\theta, \tag{A.6a}$$

$$B_{[\rho,\theta]} = 2(\cot\theta)\gamma_\rho + 2\rho^{-1}\gamma_\theta, \tag{A.6b}$$

$$C_{[\rho,\theta]} = -2\rho^2\gamma_{\rho\rho} - 2\gamma_{\theta\theta} - 4\rho\gamma_\rho + 2(\cot\theta)\gamma_\theta; \tag{A.6c}$$

$$A_{[x,y]} = -2\gamma_{xx} - 2(x^2-1)^{-1}[(1-y^2)\gamma_{yy} - 2y\gamma_y], \tag{A.7a}$$

$$B_{[x,y]} = -2y(1-y^2)^{-1}\gamma_x + 2x(x^2-1)^{-1}\gamma_y, \tag{A.7b}$$

$$C_{[x,y]} = -2\gamma_{yy} - 2(1-y^2)^{-1}[(x^2-1)\gamma_{xx} + 2x\gamma_x]; \tag{A.7c}$$

$$A_{[\nu,\eta]} = -2\gamma_{\nu\nu} - \frac{8\eta(1+\eta)^2}{(1-\nu^2)^2}\gamma_{\eta\eta} - \frac{4\eta\nu}{1+\eta\nu^2}\gamma_\nu - \frac{8(1+\eta)(1+2\eta+\eta^2\nu^2)}{(1+\eta\nu^2)(1-\nu^2)^2}\gamma_\eta, \tag{A.8a}$$

$$B_{[\nu,\eta]} = \frac{1-\eta\nu^2}{\eta(1+\eta\nu^2)}\gamma_\nu - \frac{4(1+\eta)\nu}{(1-\nu^2)(1+\eta\nu^2)}\gamma_\eta, \tag{A.8b}$$

$$C_{[\nu,\eta]} = -\frac{(1-\nu^2)^2}{2\eta(1+\eta)^2}\gamma_{\nu\nu} - 2\gamma_{\eta\eta} + \frac{\nu(1-\nu^2)(2+\eta+\eta\nu^2)}{\eta(1+\eta)^2(1+\eta\nu^2)}\gamma_\nu - \frac{2(1+(1+2\eta)\nu^2)}{(1+\eta)(1+\eta\nu^2)}\gamma_\eta; \tag{A.8c}$$

$$A_{[\sigma,\tau]} = -2\gamma_{\sigma\sigma} - 2\sigma^{-1}\tau\gamma_{\tau\tau} - 2\sigma^{-1}\gamma_\tau, \tag{A.9a}$$

$$B_{[\sigma,\tau]} = \tau^{-1}\gamma_\sigma + \sigma^{-1}\gamma_\tau, \tag{A.9b}$$

$$C_{[\sigma,\tau]} = -2\sigma\tau^{-1}\gamma_{\sigma\sigma} - 2\gamma_{\tau\tau} - 2\tau^{-1}\gamma_\sigma; \tag{A.9c}$$

$$A_{[\alpha,\beta]} = 4(\alpha+\beta)^{-1}\gamma_\alpha, \tag{A.10a}$$

$$B_{[\alpha,\beta]} = -4\gamma_{\alpha\beta}, \tag{A.10b}$$

$$C_{[\alpha,\beta]} = 4(\alpha+\beta)^{-1}\gamma_\beta. \tag{A.10c}$$

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